returning step by step to M, we see that M itself can be colored with five colors. This completes the proof. Note that this proof is constructive, in that it gives a perfectly practicable, although wearisome, method of actually coloring any map with n regions in a finite number of steps.

2. The Jordan Curve Theorem for Polygons

The Jordan curve theorem states that any simple closed curve C divides the points of the plane not on C into two distinct domains (with no points in common) of which C is the common boundary. We shall give a proof of this theorem for the case where C is a closed *polygon* P.

We shall show that the points of the plane not on P fall into two classes, A and B, such that any two points of the same class can be joined by a polygonal path which does not cross P, while any path joining a point of A to a point of B must cross P. The class A will form the "outside" of the polygon, while the class B will form the "inside."

We begin the proof by choosing a fixed direction in the plane, not parallel to any of the sides of P. Since P has but a finite number of sides, this is always possible. We now define the classes A and B as follows:

The point p belongs to A if the ray through p in the fixed direction intersects P in an *even* number, 0, 2, 4, 6, $\cdots$, of points. The point p belongs to B if the ray through p in the fixed direction intersects P in an *odd* number, 1, 3, 5, $\cdots$, of points.

With regard to rays that intersect P at vertices, we shall not count an intersection at a vertex where both edges of P meeting at the vertex are on the same side of the ray, but we shall count an intersection at a vertex where the two edges are on opposite sides of the ray. We shall say that two points p and q have the same "parity" if they belong to the same class, A or B.

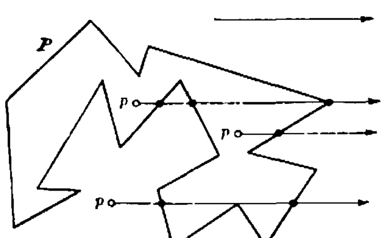

Fig. 148. Counting intersections.

First we observe that all the points on any line segment not intersecting P have the same parity. For the parity of a point p moving along such a segment can only change when the ray in the fixed direction through p passes through a vertex of P, and in neither of the two possible cases will the parity actually change, because of the agreement made in the preceding paragraph. From this it follows that *if any point* p_1 *of* A *is joined to a point* p_2 *of* B *by a polygonal path, then this path must intersect* P, for otherwise the parity of all the points of the path, and in particular of p_1 and p_2, would be the same. Moreover, we can show that *any two points of the same class,* A *or* B, *can be joined by a polygonal path which does not intersect* P. Call the two points p and q. If the straight segment pq joining p to q does not intersect P it is the desired path. Otherwise, let p' be the first point of intersection of this segment with P, and let q' be the last such point (Fig. 149). Construct the path starting from p along the segment pp', then turning off just before p' and following along P until P returns to pq at q'. If we can prove that this path will intersect pq between q' and q, rather than between p' and q', then the path may be continued to q along $q'q$ without intersecting P. It is clear that any two points r and s near enough to each other, but on opposite sides of some segment of P, must have different parity, for the ray through r will intersect P in one more point than will the ray through s. Thus we see that the parity changes as we cross the point q' along the segment pq. It follows that the dotted path crosses pq between q' and q, since p and q (and hence every point on the dotted path) have the same parity.

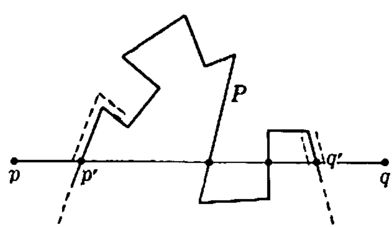

Fig. 149.

This completes the proof of the Jordan curve theorem for the case of a polygon P. The "outside" of P may now be identified as the class A, since if we travel far enough along any ray in the fixed direction we shall come to a point beyond which there will be no intersection with P, so that all such points have parity 0, and hence belong to A. This leaves the "inside" of P identified with the class B. No matter how twisted the simple closed polygon P, we can always determine whether a given point p of the plane is inside or outside P by drawing a ray and counting the number of intersections of the ray with P. If this number is odd, then the point p is imprisoned within P, and cannot escape without crossing P at some point. If the number is even, then the point p is outside P. (Try this for Figure 128.)

*One may also prove the Jordan curve theorem for polygons in the following way: Define the *order* of a point p_0 with respect to any closed curve C which does not pass through p_0 as the net number of complete revolutions made by an arrow joining p_0 to a moving point p on the curve as p traverses the curve once. Let

A = all points p_0 not on P and with *even* order with respect to P,

B = all points p_0 not on P and with *odd* order with respect to P.

Then A and B, thus defined, form the outside and inside of P respectively. The carrying out of the details of this proof is left as an exercise.

**3. The Fundamental Theorem of Algebra

The "fundamental theorem of algebra" states that if

$$f(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z + a_0, \tag{1}$$

where $n \geq 1$, and $a_{n-1}, a_{n-2}, \cdots, a_0$ are any complex numbers, then there exists a complex number α such that $f(\alpha) = 0$. In other words, *in the field of complex numbers every polynomial equation has a root.* (On p. 102 we drew the conclusion that $f(z)$ can be factored into n linear factors:

$$f(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the zeros of $f(z)$.) It is remarkable that this theorem can be proved by considerations of a topological character, related to those used in proving the Brouwer fixed point theorem.

The reader will recall that a complex number is a symbol $x + yi$, where x and y are real numbers and i has the property that $i^2 = -1$. The complex number $x + yi$ may be represented by the point in the plane whose coördinates with respect to a pair of perpendicular axes are x, y. If we introduce polar coördinates in this plane, taking the origin and the positive direction of the x-axis as pole and prime direction respectively, we may write

$$z = x + yi = r(\cos\theta + i\sin\theta),$$

where $r = \sqrt{x^2 + y^2}$. It follows from De Moivre's formula that

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$