Geometric Langlands correspondance

IMPA

20/04/2018

Local vs. Global Langlands correspondance

F is a local non-arquimedean field like $\mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$.

$$\boxed{\begin{array}{l}\text{Irreducible rank n}\\ \text{reps. of Gal}(\bar{F}/\text{F})\end{array}} \leftrightarrow \boxed{\begin{array}{l}\text{Irreducible smooth}\\ \text{reps of } GL_n(\text{F})\end{array}}$$

F a number field or a function field of a curve defined over $\mathbb{F}_q$

$$\boxed{\begin{array}{l}\text{Irreducible rank n}\\ \text{reps. of Gal}(\bar{F}/\text{F})\end{array}} \leftrightarrow \boxed{\begin{array}{l}\text{Automorphic}\\ \text{reps of } GL_n(\mathbb{A}_F)\end{array}}$$

Separable Extensions

$F = \mathbb{F}_q((t))$. $\mathbb{F}_q((t^{1/p}))$ is non-separable.

$$x^p - (t^{1/p})^p = (x - t^{1/p})^p.$$

$\bar{F}$ is the maximal separable extension of F inside its algebraic closure.

$$Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q) \simeq \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}.$$

generated by Frobenius.

Weil and Weil-Deligne groups

$$\pi : Gal(\overline{F}/F) \to Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q). \qquad W_F = \pi^1(\mathbb{Z})$$

$$\tilde{W}_F = W_F \ltimes \mathbb{C}$$

$$\sigma x \sigma^{-1} = q^{\pi(\sigma)} x, \qquad \sigma \in W_F, x \in \mathbb{C}$$

$$\boxed{\text{Admissible reps of } \tilde{W}_F} \leftrightarrow \boxed{\ell\text{-adic reps of } W_F.}$$

Local Langlands correspondance

F is a local non-arquimedean field like $\mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$.

Irreducible rank n reps. of $\tilde{W}_F$	$\leftrightarrow$	Irreducible smooth reps of GL_n(F)

Other reductive groups

$Gal(\overline{F}/F)$ acts on root datum for G defined over F.

${}^L G$ is defined over $\mathbb{C}$.

$$\boxed{W_F \to {}^L G} \leftrightarrow \boxed{\text{Irreducible smooth reps of G(F)}}$$

Global Langlands correspondance

$F = \mathbb{F}_q(X)$ or $\mathbb{Q} \subset F \subset \mathbb{C}$.

$$F \hookrightarrow \mathbb{A}_F = \prod_{x \in X}{}' F_x.$$

Automorphic means that it is realized in $GL_n(F) \backslash GL_n(\mathbb{A}_F)$.

$$\boxed{\begin{array}{l}\text{Irreducible rank n} \\ \ell\text{-adic reps. of } W_F\end{array}} \leftrightarrow \boxed{\begin{array}{l}\text{Automorphic} \\ \text{reps of } GL_n(\mathbb{A}_F)\end{array}}$$

Frobenius eigenvalues $\leftrightarrow$ Hecke eigenvalues

Local to global

$x \in X$, $F_x \simeq \mathbb{F}_{q_x}((t))$ the local field and $\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t]]$ the ring of integers.

$W_{F_x} \hookrightarrow W_F$. So a representation σ of W_F induces a representation σ_x of W_{F_x}.

Use local Langlands correspondance to obtain π_x a representation of $GL_n(F_x)$.

$$\pi_\sigma = {\bigotimes_{x \in X}}' \pi_x.$$

The spherical Hecke algebra

$\mathcal{H}_x$ is the algebra of compactly supported functions on

$$GL_n(\mathcal{O}_x)\backslash GL_n(F_x)/GL_n(\mathcal{O}_x).$$

it acts on π_x by

$$f_x \star v = \int f_x(g)(g \cdot v)dg, \qquad f_x \in \mathcal{H}_x, v \in \pi_x$$

If π_x is irreducible we obtain a character of $\mathcal{H}_x$.

Theorem (Strong multiplicity I. Piatetski-Shapiro) the collection of characters determines π up to an isomorphism.

Number fields

The completions of $\mathbb{Q}$ are either $\mathbb{Q}_p$ or $\mathbb{R}$.

Some Automorphic representations do not correspond to Galois reps.

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on geometric invariants like étale cohomologies of varieties defined over $\mathbb{Q}$. For E an elliptic curve, we have a two-dimensional representation.

Invariants on Galois side, like number of points in $E(\mathbb{F}_q)$ are translated to modular invariants on the $G(\mathbb{A}_{\mathbb{Q}})$ side.

Local systems

X a curve defined over k and $F = k(X)$, then $Gal(\overline{F}/F)$ is the "*fundamental group*" of X.

$$\boxed{\pi^1(X, x_0) \to GL_n(\mathbb{C})} \leftrightarrow \boxed{\text{rank n local systems on X}}$$

$$\boxed{\begin{array}{l}\text{rank n holomorphic vector bundles}\\ \text{with connection on X}\end{array}} \leftrightarrow \boxed{\text{rank n local systems on X}}$$

Automorphic functions from representations

$F = \mathbb{F}_q(X)$.

$$\boxed{\begin{array}{l}\text{(unramified) automorphic} \\ \text{reps. of } GL_n(\mathbb{A}_F)\end{array}} \to \boxed{\begin{array}{l}GL_n(\mathcal{O})\text{-invariant functions of} \\ GL_n(F)\backslash GL_n(\mathbb{A})\end{array}}$$

Lemma(Weil)

$$GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$$

is in bijection with the set of isomorphism classes of vector bundles on X.

Sheaves on Bun_n

$$\boxed{\begin{array}{l}\text{Functions on}\\ GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})\end{array}} \leftrightarrow \boxed{\text{sheaves on } Bun_n(X)}$$

Fonctions-faisceaux correspondance:

- Need to use ℓ-adic sheaves.
- For $\mathcal{F}$ an ℓ-adic sheaf on X and $x \in X$ an $\mathbb{F}_q$-point. $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on $\mathcal{F}_{\bar{x}}$.
- Taking the trace of Frobenius (which is a generator of this group) obtain a function with values on $\overline{\mathbb{Q}}_\ell$.

Hecke correspondances

$$\mathcal{H}ecke_n = \{\phi : \mathcal{P}' \hookrightarrow \mathcal{P}, x \in X\}$$

$$0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{O}_x^i \to 0, \qquad \mathcal{P}', \mathcal{P} \in Bun_n$$

$$\begin{array}{ccc} & \mathcal{H}ecke_n & \\ {}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}} \\ Bun_n & & X \times Bun_n \end{array}$$

$h^{\leftarrow}(\phi, x) = \mathcal{P}$, $h^{\rightarrow}(\phi, x) = (x, \mathcal{P}')$.

Geometric Langlands correspondance

$$H(\mathcal{F}) = h_*^{\rightarrow} h^{\leftarrow *}(\mathcal{F}).$$

Let E be a local system of rank n on X. A Hecke eigensheaf with eigenvalue E is:

$$H_n(\mathcal{F}) \simeq E \boxtimes \mathcal{F}$$

Geometric Langlands correspondance (Deligne, Drinfeld-Laumon, Frenkel-Gaitsgory-Vilonen)

$$\boxed{\begin{array}{l}\text{Irreducible rank n}\\ \text{local systems on X}\end{array}} \leftrightarrow \boxed{\text{Hecke eigensheaves on } Bun_n(X)}$$

Geometric Langlands correspondance

$$H(\mathcal{F}) = h_*^{\rightarrow} h^{\leftarrow *}(\mathcal{F}).$$

Let E be a local system of rank n on X. A Hecke eigensheaf with eigenvalue E is:

$$H_n^i(\mathcal{F}) \simeq \wedge^i E \boxtimes \mathcal{F}[-i(n-i)], \qquad i = 1, \cdots, n.$$

Geometric Langlands correspondance (Deligne, Drinfeld-Laumon, Frenkel-Gaitsgory-Vilonen)

$$\boxed{\begin{array}{l}\text{Irreducible rank n} \\ \text{local systems on X}\end{array}} \leftrightarrow \boxed{\text{Hecke eigensheaves on } Bun_n(X)}$$

Geometric class field theory

Let E be a rank one local system on X, so it is a morphism $H_1(X, \mathbb{Z}) \simeq \pi^1(Jac) \to \mathbb{C}^\times$.

This produces a local system on $Jac(X)$ which is a Hecke eigensheaf.

The local setting

When $F = \mathbb{C}((t))$, $G(F)$ is known as the *loop group* of G.

$$Gal(\overline{F}/F) \simeq \hat{\mathbb{Z}}.$$

There are few representations!

$$Gal(\overline{F}/F) \to {}^L G.$$

Flat bundles

When X is not compact, there are more flat bundles than local systems.

$$\boxed{\begin{array}{l}\text{Holomorphic vector bundles}\\ \text{on } X\setminus S \text{ with regular singularities}\end{array}} \leftrightarrow \boxed{\text{Local systems on } X\setminus S}$$

On $X = Spec\mathbb{C}((t))$, Holomorphic vector bundles are in correspondance with operators

$$\nabla_{\partial t} = \partial_t + A(t), \qquad A(t) \in \mathfrak{gl}_n\big(\mathbb{C}((t))\big).$$

modulo gauge transformations:

$$A(t) \mapsto gA(t)g^{-1} - (\partial_t g)g^{-1}, \qquad g \in GL_n((t)).$$

Representation side

There are too few algebraic representations of $G((t))$

Theorem A smooth integrable representation of $\mathfrak{g}((t))$ for $\mathfrak{g}$ semisimple, is trivial.

As in the global case, representations of $G(F)$, give rise to locally compact functions on $G(F)/G(\mathcal{O})$, the *affine Grassmanian*.

When $F = \mathbb{C}((t))$ this space is an infinite dimensional *ind-scheme*.

Frenkel-Gaitsgory proposal

Flat connections on Spec $\mathbb{C}((t))$ modulo gauge transformations	$\leftrightarrow$	Representations of $G((t))$ on sheaves of categories on the affine grassmanian