

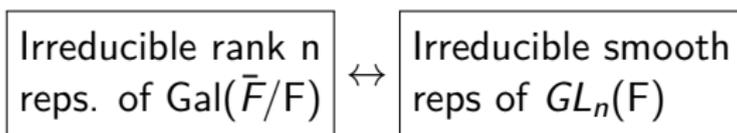
Geometric Langlands correspondance

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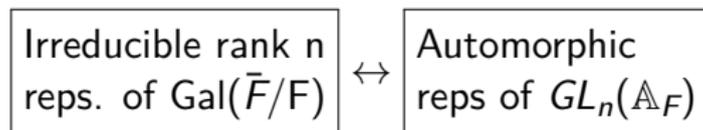
20/04/2018

Local vs. Global Langlands correspondance

F is a local non-archimedean field like $\mathbb{F}_q((t))$ or a finite extension of \mathbb{Q}_p .



F a number field or a function field of a curve defined over \mathbb{F}_q



Separable Extensions

$F = \mathbb{F}_q((t))$. $\mathbb{F}_q((t^{1/p}))$ is non-separable.

$$x^p - (t^{1/p})^p = (x - t^{1/p})^p.$$

\bar{F} is the maximal separable extension of F inside its algebraic closure.

$$\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \simeq \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}.$$

generated by Frobenius.

Weil and Weil-Deligne groups

$$\pi : \text{Gal}(\overline{F}/F) \rightarrow \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q). \quad W_F = \pi^{-1}(\mathbb{Z})$$

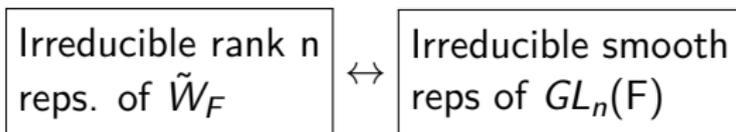
$$\tilde{W}_F = W_F \rtimes \mathbb{C}$$

$$\sigma x \sigma^{-1} = q^{\pi(\sigma)} x, \quad \sigma \in W_F, x \in \mathbb{C}$$

$$\boxed{\text{Admissible reps of } \tilde{W}_F} \leftrightarrow \boxed{\ell\text{-adic reps of } W_F.}$$

Local Langlands correspondance

F is a local non-archimedean field like $\mathbb{F}_q((t))$ or a finite extension of \mathbb{Q}_p .



Other reductive groups

$\text{Gal}(\bar{F}/F)$ acts on root datum for G defined over F .

${}^L G$ is defined over \mathbb{C} .

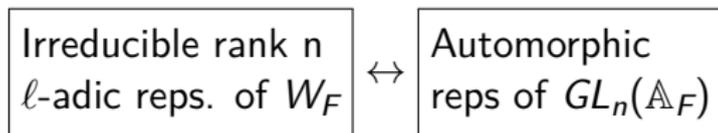
$$\boxed{W_F \rightarrow {}^L G} \leftrightarrow \boxed{\text{Irreducible smooth reps of } G(F)}$$

Global Langlands correspondance

$F = \mathbb{F}_q(X)$ or $\mathbb{Q} \subset F \subset \mathbb{C}$.

$$F \hookrightarrow \mathbb{A}_F = \prod'_{x \in X} F_x.$$

Automorphic means that it is realized in $GL_n(F) \backslash GL_n(\mathbb{A}_F)$.



Frobenius eigenvalues \leftrightarrow Hecke eigenvalues

Local to global

$x \in X$, $F_x \simeq \mathbb{F}_{q_x}((t))$ the local field and $\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t]]$ the ring of integers.

$W_{F_x} \hookrightarrow W_F$. So a representation σ of W_F induces a representation σ_x of W_{F_x} .

Use local Langlands correspondance to obtain π_x a representation of $GL_n(F_x)$.

$$\pi_\sigma = \bigotimes'_{x \in X} \pi_x.$$

The spherical Hecke algebra

\mathcal{H}_x is the algebra of compactly supported functions on

$$GL_n(\mathcal{O}_x) \backslash GL_n(F_x) / GL_n(\mathcal{O}_x).$$

it acts on π_x by

$$f_x \star v = \int f_x(g)(g \cdot v) dg, \quad f_x \in \mathcal{H}_x, v \in \pi_x$$

If π_x is irreducible we obtain a character of \mathcal{H}_x .

Theorem (Strong multiplicity I. Piatetski-Shapiro) the collection of characters determines π up to an isomorphism.

Number fields

The completions of \mathbb{Q} are either \mathbb{Q}_p or \mathbb{R} .

Some Automorphic representations do not correspond to Galois reps.

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on geometric invariants like étale cohomologies of varieties defined over \mathbb{Q} . For E an elliptic curve, we have a two-dimensional representation.

Invariants on Galois side, like number of points in $E(\mathbb{F}_q)$ are translated to modular invariants on the $G(\mathbb{A}_{\mathbb{Q}})$ side.

Local systems

X a curve defined over k and $F = k(X)$, then $\text{Gal}(\bar{F}/F)$ is the “*fundamental group*” of X .

$$\boxed{\pi^1(X, x_0) \rightarrow GL_n(\mathbb{C})} \leftrightarrow \boxed{\text{rank } n \text{ local systems on } X}$$

$$\boxed{\text{rank } n \text{ holomorphic vector bundles with connection on } X} \leftrightarrow \boxed{\text{rank } n \text{ local systems on } X}$$

Automorphic functions from representations

$$F = \mathbb{F}_q(X).$$



Lemma(Weil)

$$GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$$

is in bijection with the set of isomorphism classes of vector bundles on X .

Sheaves on Bun_n

$$\boxed{\text{Functions on } GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})} \leftrightarrow \boxed{\text{sheaves on } Bun_n(X)}$$

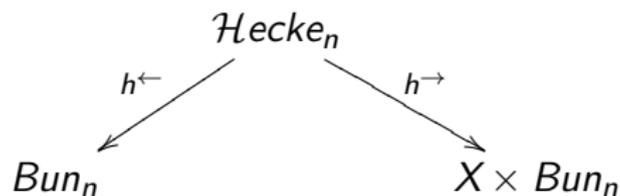
Fonctions-faisceaux correspondance:

- ▶ Need to use ℓ -adic sheaves.
- ▶ For \mathcal{F} an ℓ -adic sheaf on X and $x \in X$ an \mathbb{F}_q -point. $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on $\mathcal{F}_{\bar{x}}$.
- ▶ Taking the trace of Frobenius (which is a generator of this group) obtain a function with values on $\overline{\mathbb{Q}_\ell}$.

Hecke correspondances

$$\mathcal{H}ecke_n = \{ \phi : \mathcal{P}' \hookrightarrow \mathcal{P}, x \in X \}$$

$$0 \rightarrow \mathcal{P}' \rightarrow \mathcal{P} \rightarrow \mathcal{O}_x^i \rightarrow 0, \quad \mathcal{P}', \mathcal{P} \in Bun_n$$



$$h^{\leftarrow}(\phi, x) = \mathcal{P}, \quad h^{\rightarrow}(\phi, x) = (x, \mathcal{P}').$$

Geometric Langlands correspondance

$$H(\mathcal{F}) = h_*^{\rightarrow} h^{\leftarrow*}(\mathcal{F}).$$

Let E be a local system of rank n on X . A Hecke eigensheaf with eigenvalue E is:

$$H_n(\mathcal{F}) \simeq E \boxtimes \mathcal{F}$$

Geometric Langlands correspondance (Deligne, Drinfeld-Laumon, Frenkel-Gaitsgory-Vilonen)

Irreducible rank n
local systems on X

\leftrightarrow

Hecke eigensheaves on $Bun_n(X)$

Geometric Langlands correspondance

$$H(\mathcal{F}) = h_*^{\rightarrow} h^{\leftarrow*}(\mathcal{F}).$$

Let E be a local system of rank n on X . A Hecke eigensheaf with eigenvalue E is:

$$H_n^i(\mathcal{F}) \simeq \wedge^i E \boxtimes \mathcal{F}[-i(n-i)], \quad i = 1, \dots, n.$$

Geometric Langlands correspondance (Deligne, Drinfeld-Laumon, Frenkel-Gaitsgory-Vilonen)

Irreducible rank n
local systems on X

\leftrightarrow

Hecke eigensheaves on $Bun_n(X)$

Geometric class field theory

Let E be a rank one local system on X , so it is a morphism $H_1(X, \mathbb{Z}) \simeq \pi^1(\text{Jac}) \rightarrow \mathbb{C}^\times$.

This produces a local system on $\text{Jac}(X)$ which is a Hecke eigensheaf.

The local setting

When $F = \mathbb{C}((t))$, $G(F)$ is known as the *loop group* of G .

$$\mathrm{Gal}(\bar{F}/F) \simeq \hat{\mathbb{Z}}.$$

There are few representations!

$$\mathrm{Gal}(\bar{F}/F) \rightarrow {}^L G.$$

Flat bundles

When X is not compact, there are more flat bundles than local systems.

Holomorphic vector bundles
on $X \setminus S$ with regular singularities

\leftrightarrow

Local systems on $X \setminus S$

On $X = \text{Spec} \mathbb{C}((t))$, Holomorphic vector bundles are in correspondance with operators

$$\nabla_{\partial_t} = \partial_t + A(t), \quad A(t) \in \mathfrak{gl}_n(\mathbb{C}((t))).$$

modulo gauge transformations:

$$A(t) \mapsto gA(t)g^{-1} - (\partial_t g)g^{-1}, \quad g \in GL_n((t)).$$

Representation side

There are too few algebraic representations of $G((t))$

Theorem A smooth integrable representation of $\mathfrak{g}((t))$ for \mathfrak{g} semisimple, is trivial.

As in the global case, representations of $G(F)$, give rise to locally compact functions on $G(F)/G(\mathcal{O})$, the *affine Grassmanian*.

When $F = \mathbb{C}((t))$ this space is an infinite dimensional *ind-scheme*.

Frenkel-Gaitsgory proposal

Flat connections on
 $\text{Spec } \mathbb{C}((t))$
modulo gauge transformations

\leftrightarrow

Representations of $G((t))$ on
sheaves of categories on
the affine grassmanian