15th Brazilian Symposium on Computer Graphics and Image Processing

Approximating Parametric Curves with Strip Trees using Affine Arithmetic

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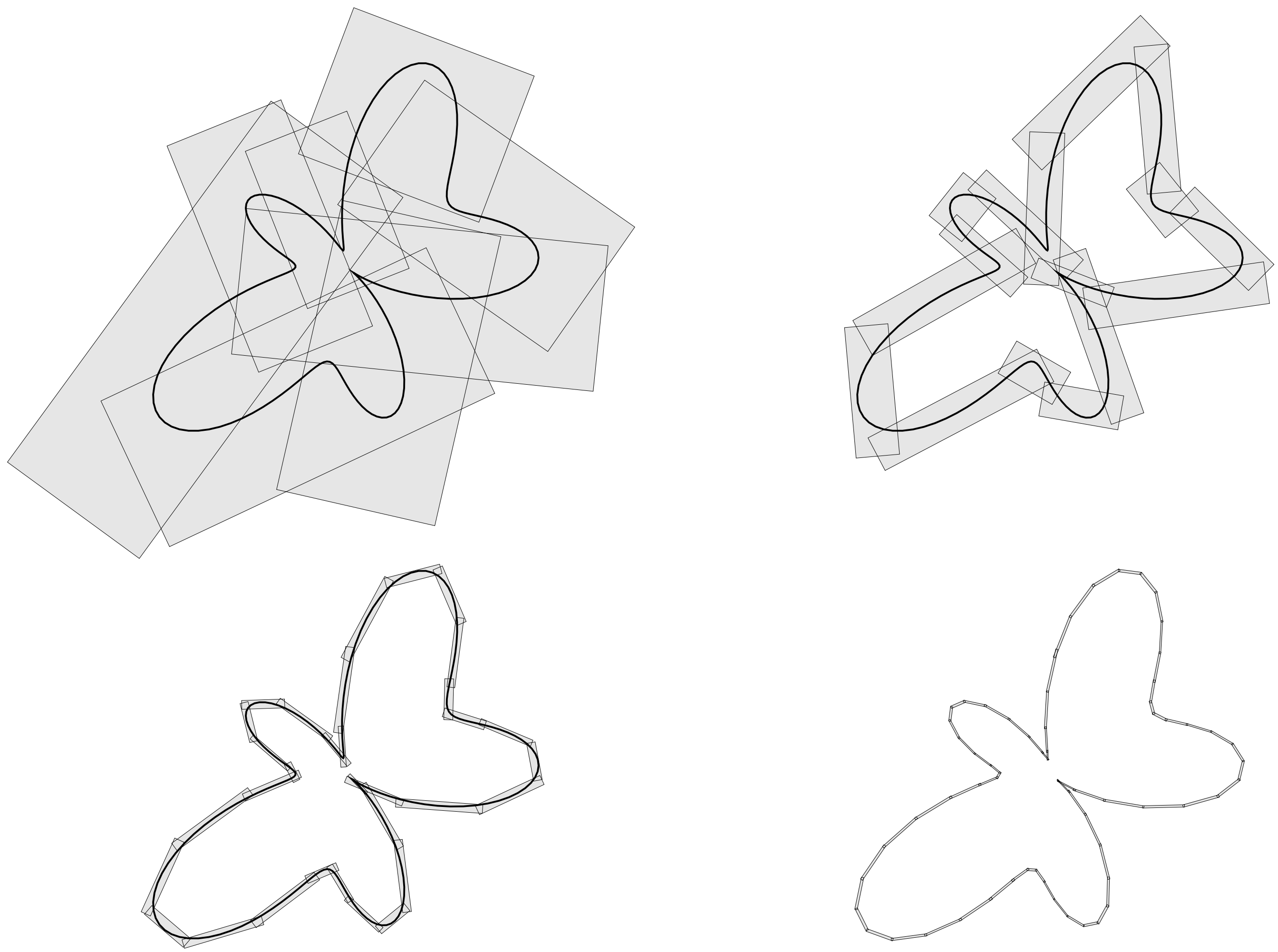
Strip trees

- Multi-resolution representation for polygonal curves (Ballard, 1981)
  - ◇ tree of rectangles enclosing pieces of the curve

- Many applications:
  - ◇ display at given resolution
  - ◇ curve intersection
  - ◇ approximate length computation
  - ◇ testing point proximity
  - ◇ testing point location

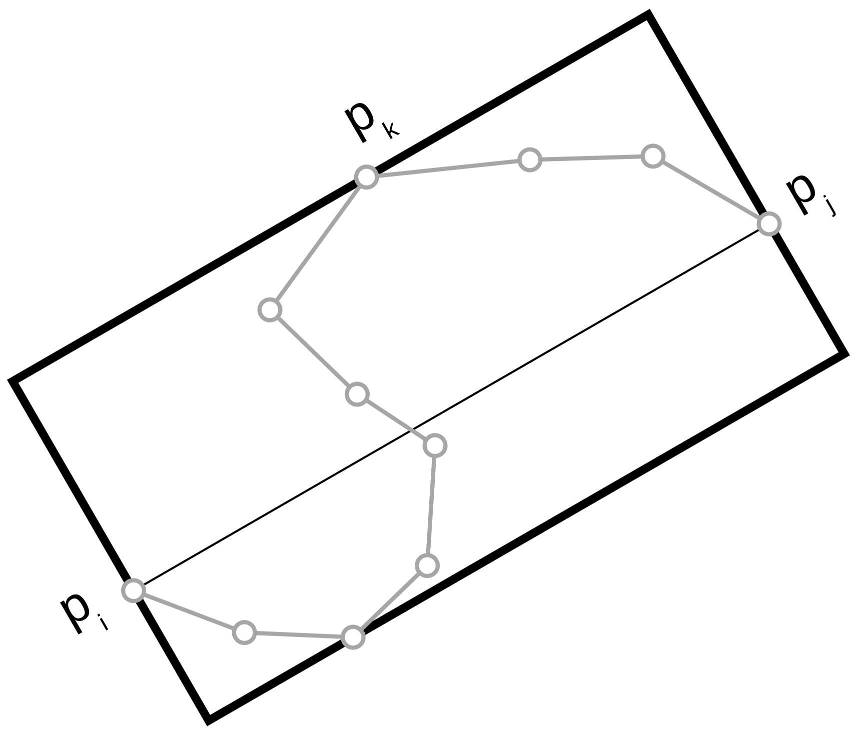
- We shall extend strip trees to general parametric curves

A strip tree



Strip trees for polygonal curves

- Start with whole curve $\mathcal{C} = p_1 \ldots p_n$
- Find bounding rectangle
- Choose *splitting point* p_k
- Recursively build strip trees for two halves $p_1 \ldots p_k$ and $p_k \ldots p_n$.



Strip trees for parametric curves

- Parametric curve $\mathcal{C} = \gamma(I)$ given by γ: $I \subseteq \mathbf{R} \to \mathbf{R}^2$
- Strip tree for $\mathcal{C}$ is the result of strip-tree(I)

```
strip-tree(T):
    B ← bounding rectangle for P = γ(T)
    if leaf(T, B) then
        return ⟨T, B, nil, nil⟩
    else
        split T into T1 and T2
        return ⟨T, B, strip-tree(T1), strip-tree(T2)⟩
```

- Crucial steps:
  - ◇ bounding rectangle: use affine arithmetic to avoid heuristics
  - ◇ split T at midpoint
  - ◇ stop recursion with application-dependent predicate (leaf)

Affine arithmetic

- Tool for validated numerics introduced in SIBGRAPI'93

- Used in robust solution of several graphics problems as a replacement for interval arithmetic

- Represents a quantity x with an *affine form*

$$\widehat{x} = x_0 + x_1\,\varepsilon_1 + \cdots + x_n\,\varepsilon_n$$

  Noise symbols $\varepsilon_i \in \mathbf{U} = [-1, +1]$, independent but otherwise unknown

- We can compute arbitrary formulas on affine forms

- Key feature: ability to handle correlations

Geometry of affine arithmetic

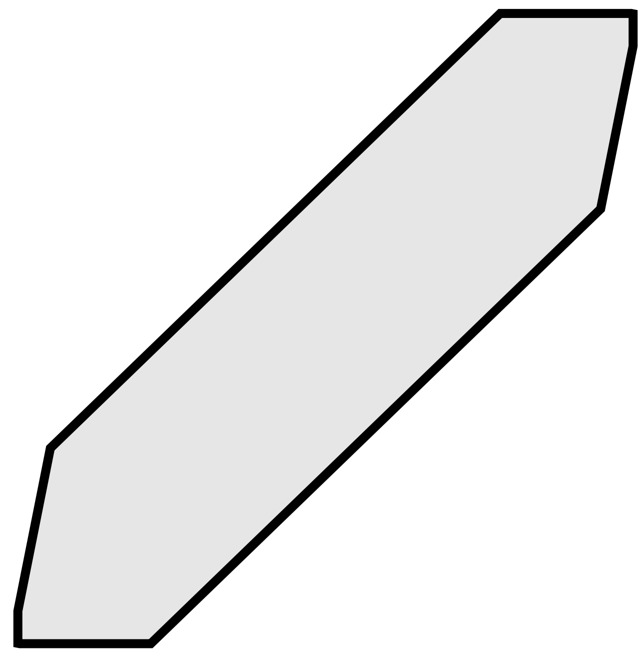
Affine forms that share noise symbols are not independent:

$$\begin{aligned} \hat{x} &= x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n \\ \hat{y} &= y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n \end{aligned}$$

The region containing (x, y) is

$$Z = \{(x, y) : \varepsilon_i \in \mathbf{U}\}$$

Z is the image of $\mathbf{U}^n$ under an affine map $\mathbf{R}^n \to \mathbf{R}^2$ and so Z is a centrally symmetric convex polygon, a *zonotope*.



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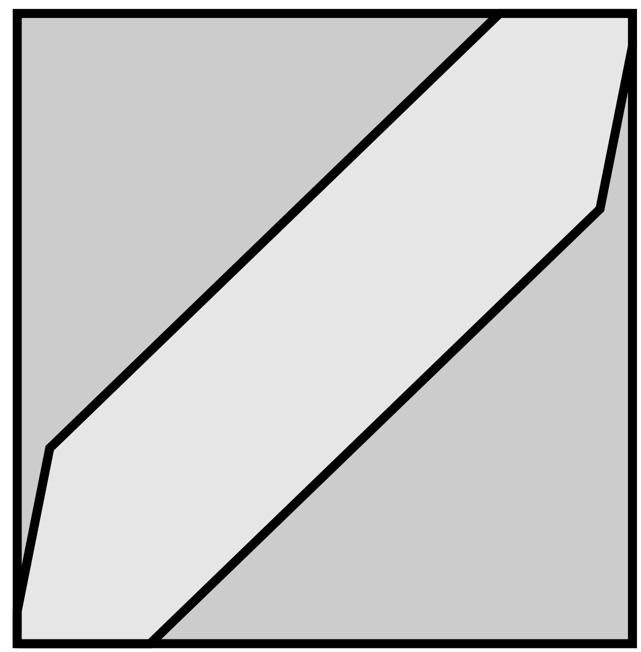
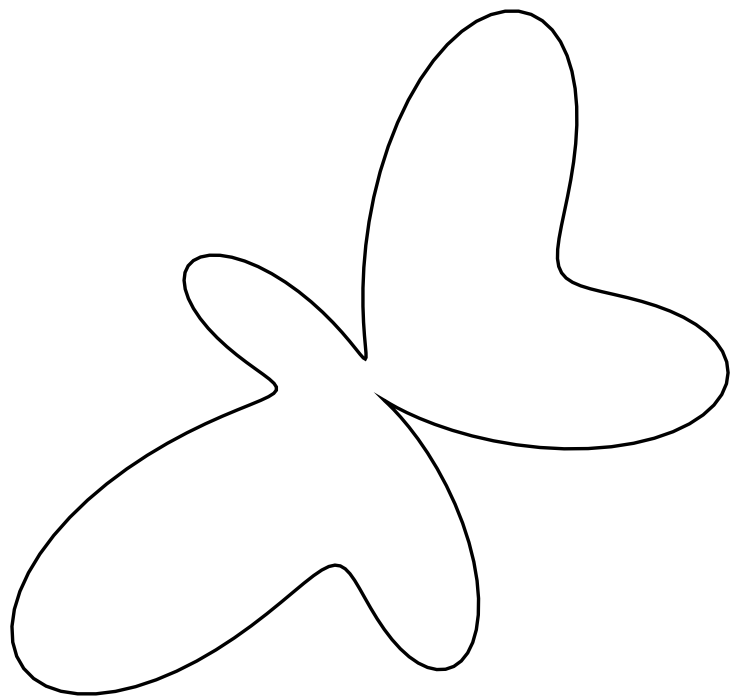
The region would be a rectangle if x and y were independent.



Approximating parametric curves

Given a parametric curve $\mathcal{C} = \gamma(I)$, where $\gamma\colon I \to \mathbf{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$.



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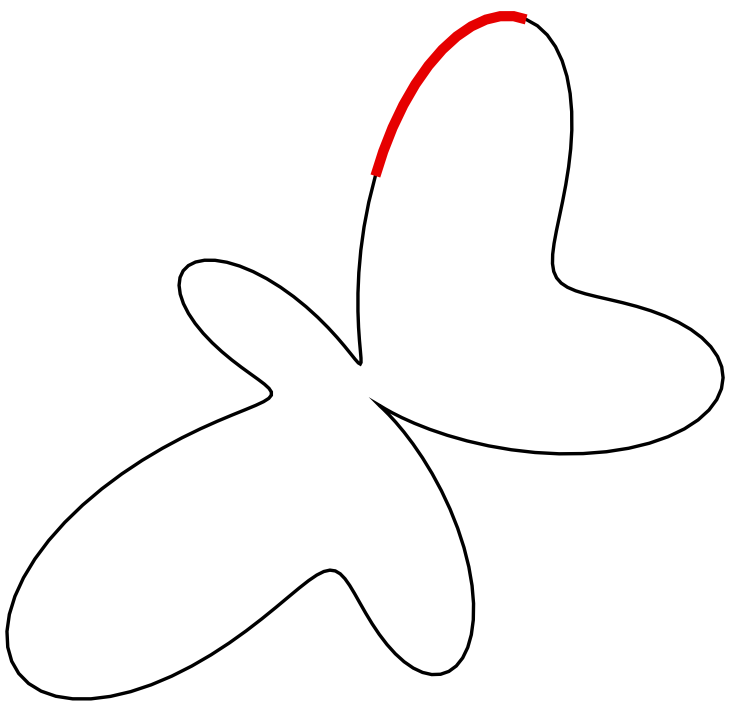

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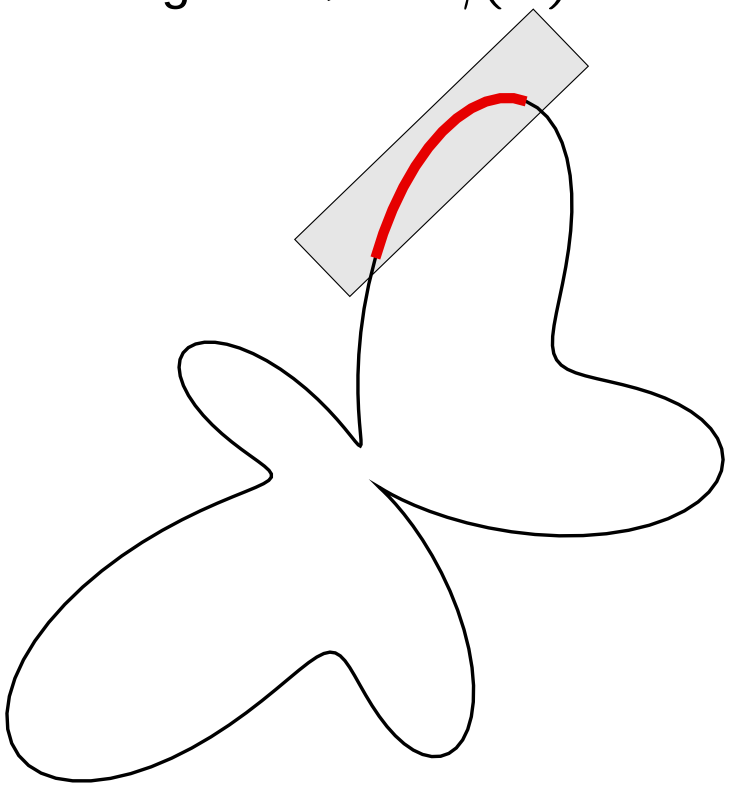

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Given a parametric curve $\mathcal{C} = \gamma(I)$, where $\gamma\colon I \to \mathbf{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$.

Solution with AA:

- Write $\gamma(t) = (x(t), y(t))$.
- Represent $t \in T$ with an affine form:

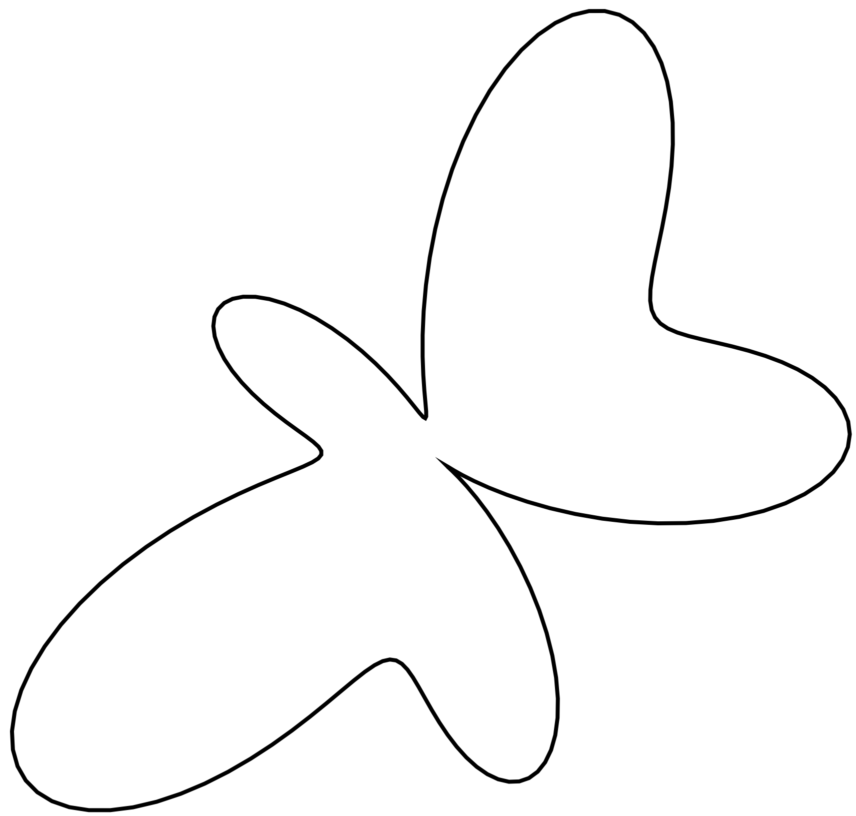
$$\hat{t} = t_0 + t_1\,\varepsilon_1, \qquad t_0 = (b+a)/2, \quad t_1 = (b-a)/2$$

- Compute coordinate functions x and y at $\hat{t}$ using AA:

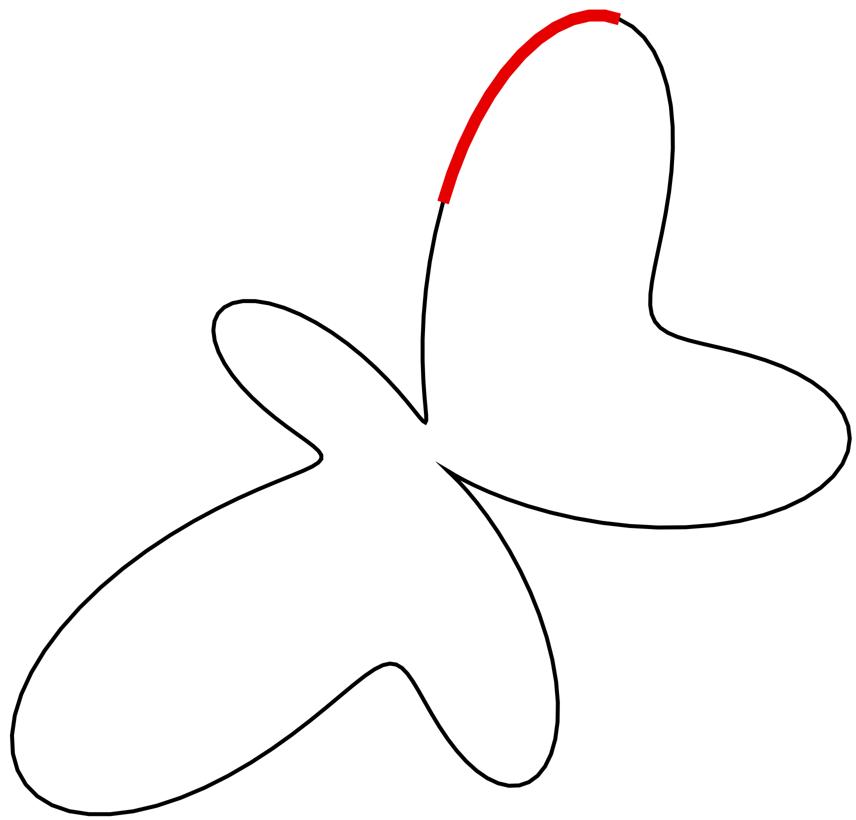
$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$
$$\hat{y} = y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n$$

- Use bounding rectangle of the xy zonotope.

Approximating parametric curves



Approximating parametric curves



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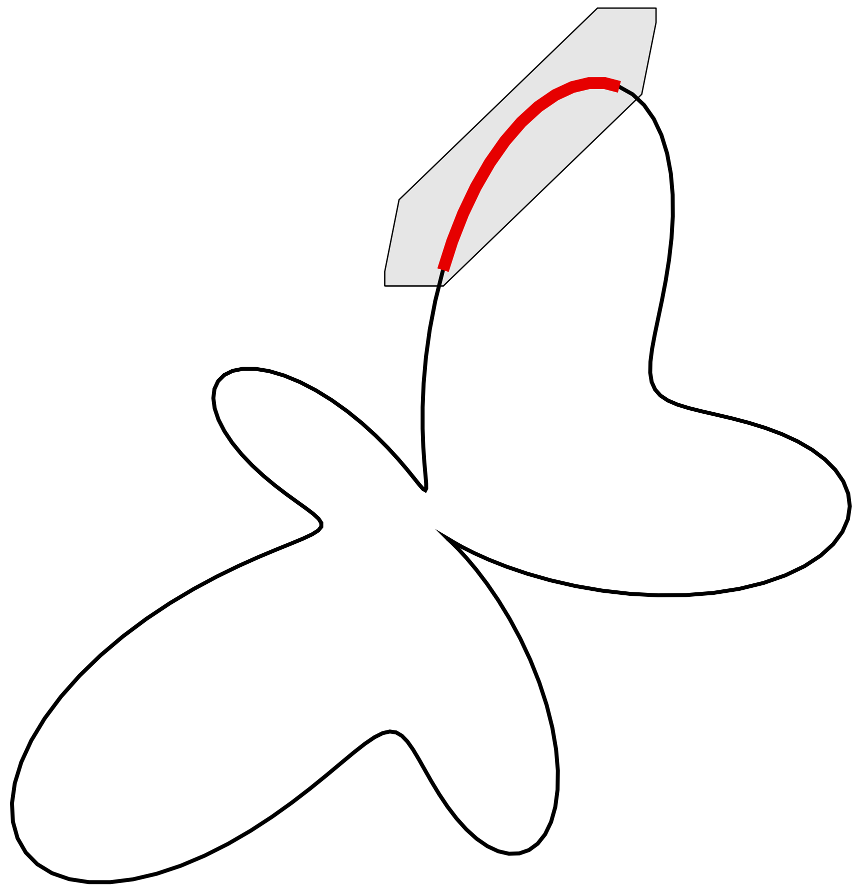

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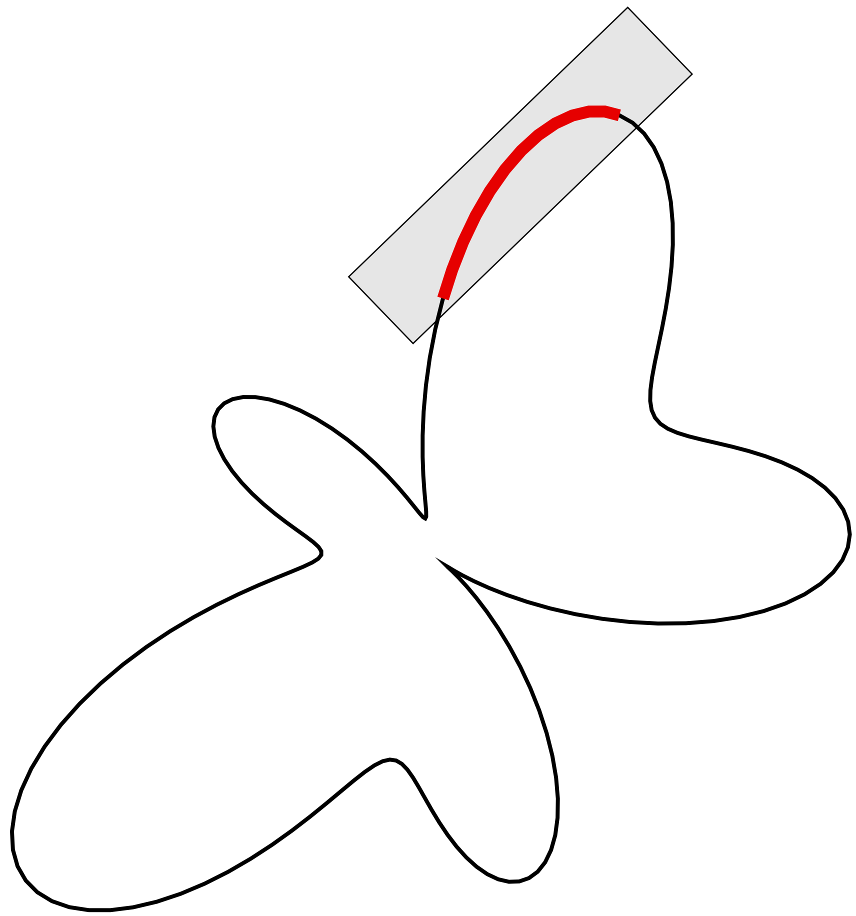
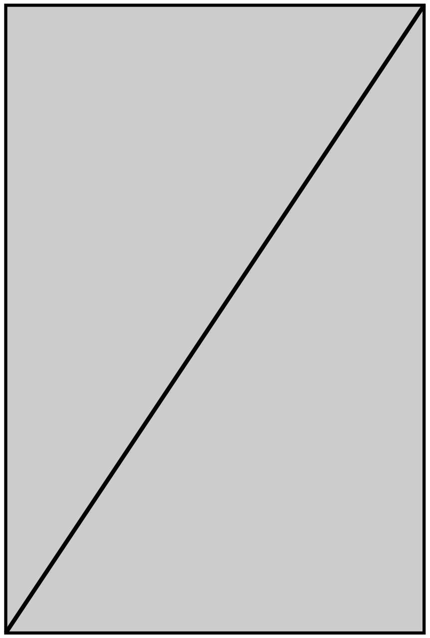


Approximating parametric curves (example)

- $\mathcal{C}$ = line segment given by $\gamma(t) = (1,1) + t(4,6)$, for $t \in [0,1]$

$$\begin{aligned} \hat{t} &= 0.5 + 0.5\,\varepsilon_1 \\ \hat{x} &= 1 + 4\,\hat{t} = 3 + 2\,\varepsilon_1 \\ \hat{y} &= 1 + 6\,\hat{t} = 4 + 3\,\varepsilon_1 \end{aligned}$$

Separately: $(x, y) \in [1,5] \times [1,7]$



Jointly: (x, y) is exactly on the line segment

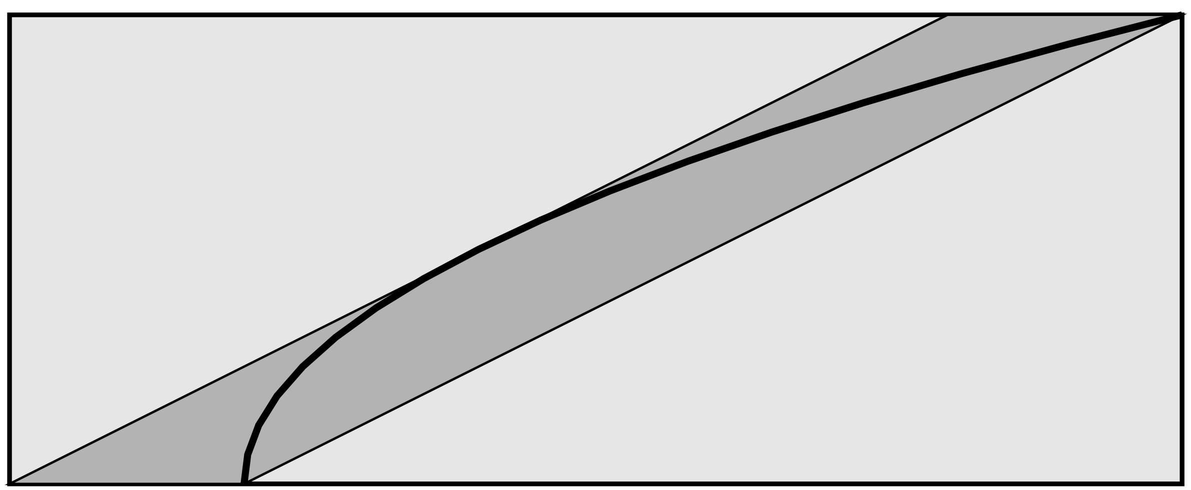
Approximating parametric curves (example)

- $\mathcal{C}$ = parabolic segment given by $\gamma(t) = (t^2, t)$, for $t \in [0, 2]$

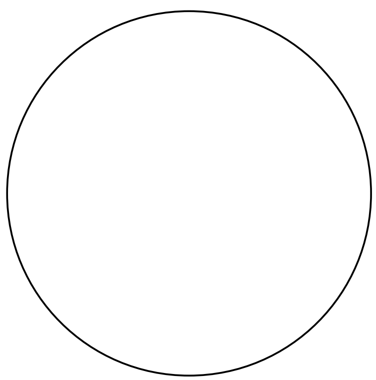
$$\hat{x} = \hat{t}^2 = 1.5 + 2\,\varepsilon_1 + 0.5\,\varepsilon_2$$
$$\hat{y} = \hat{t} \quad = 1 + 1\,\varepsilon_1$$

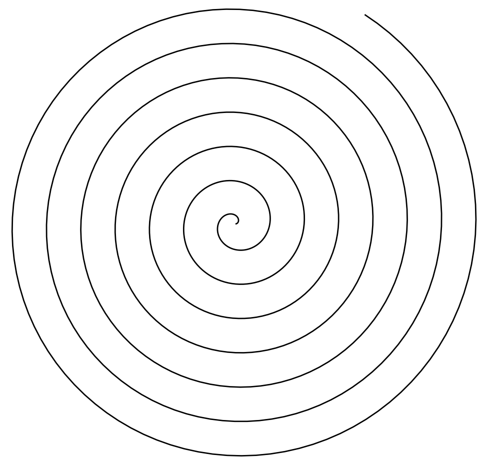
Separately: $(x, y) \in = [-1, 4] \times [0, 2]$

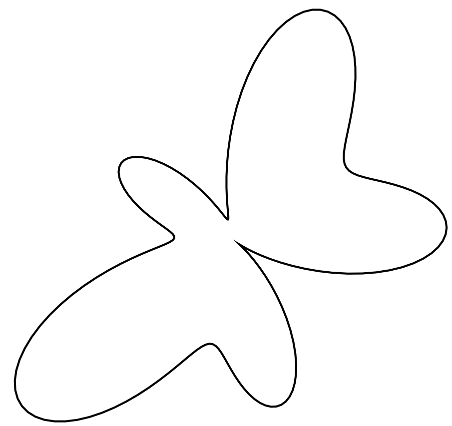
Jointly: (x, y) is in parallelogram



Examples of strip-tree approximations

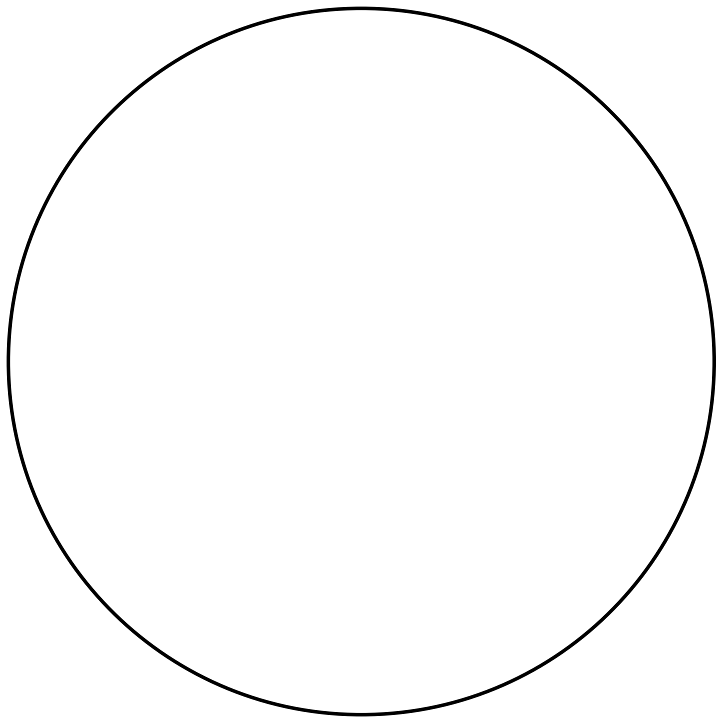




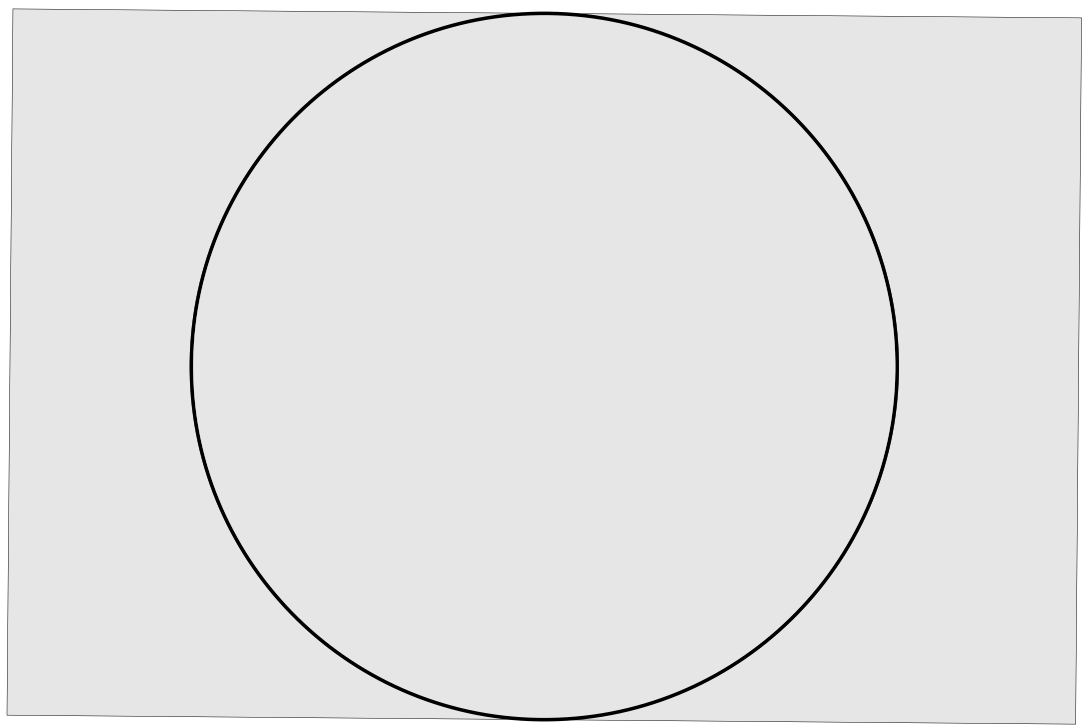


Circle Spiral Butterfly

Strip tree for circle



Strip tree for circle



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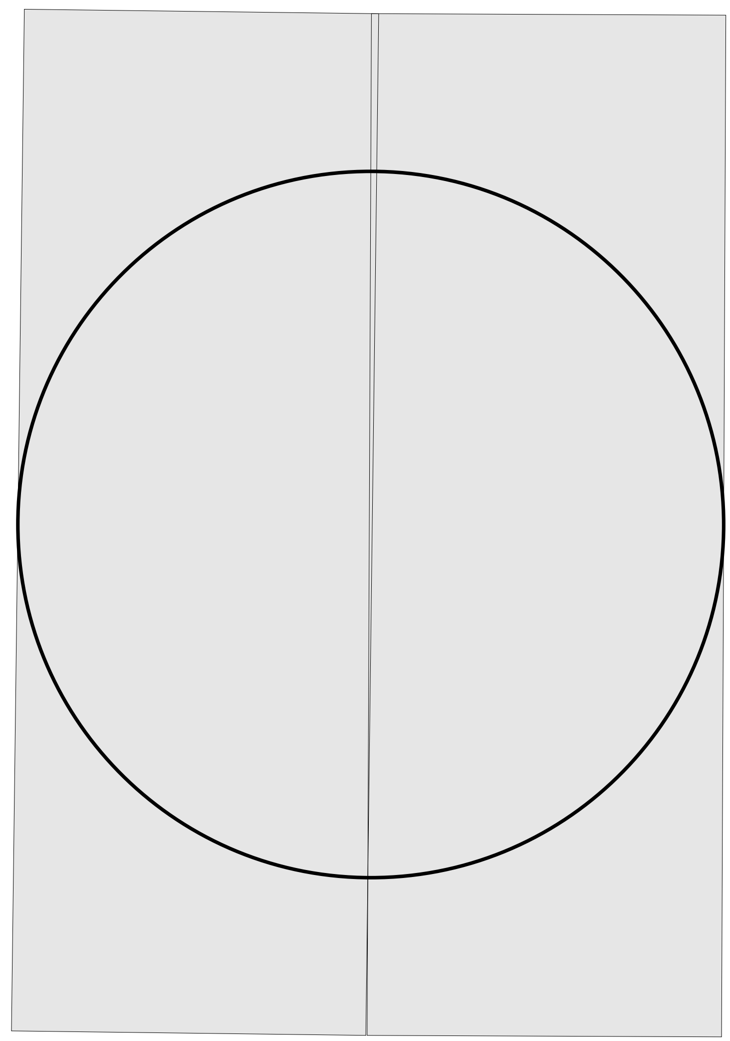

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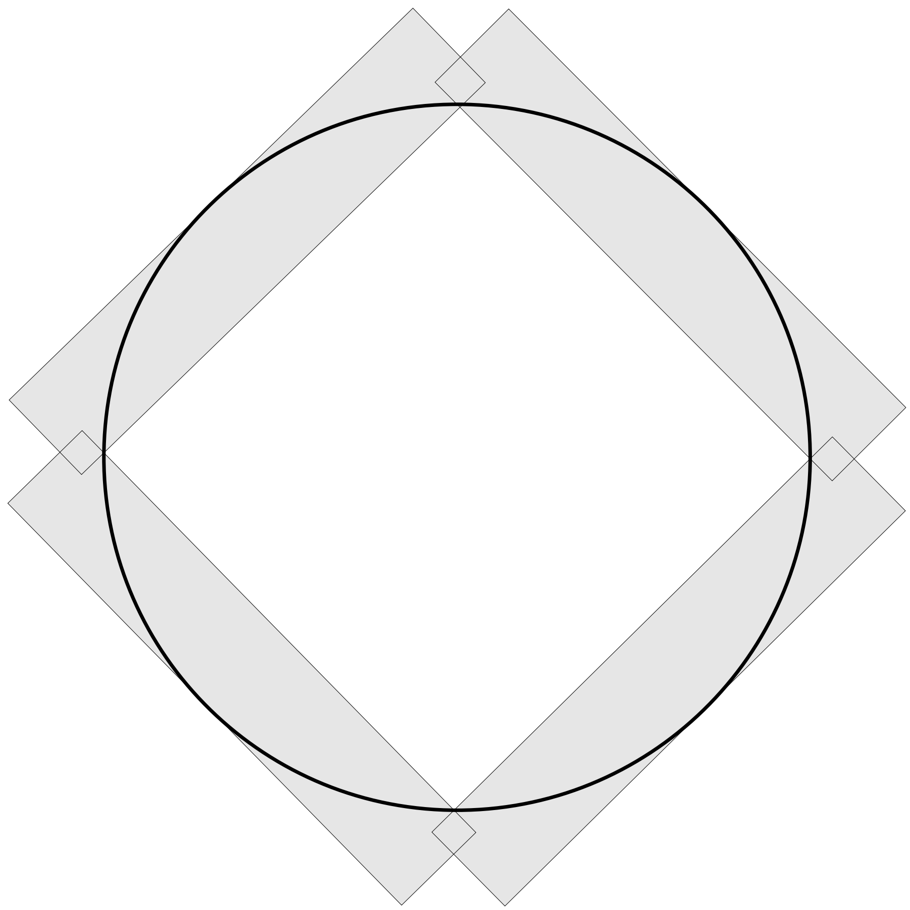
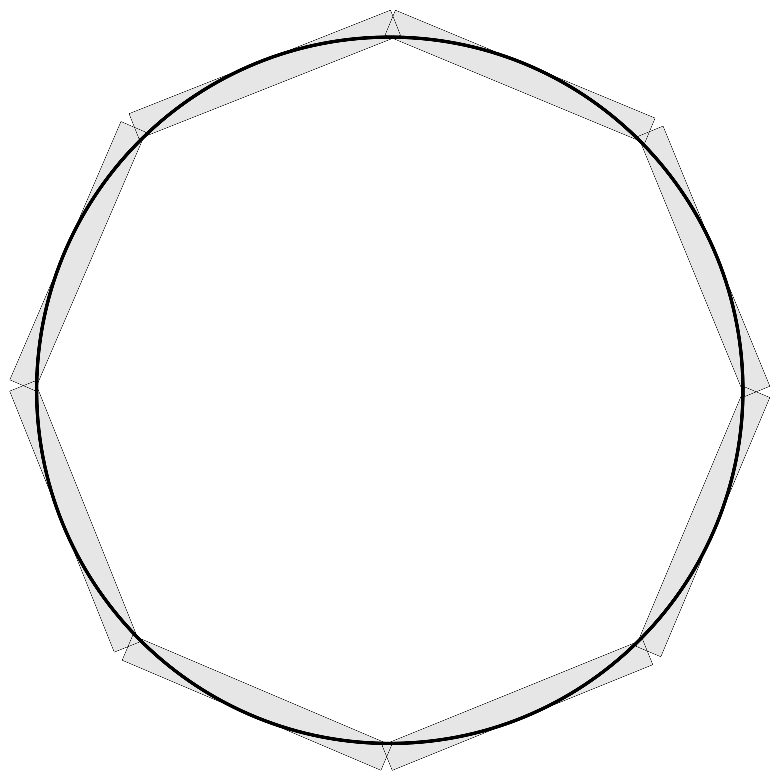
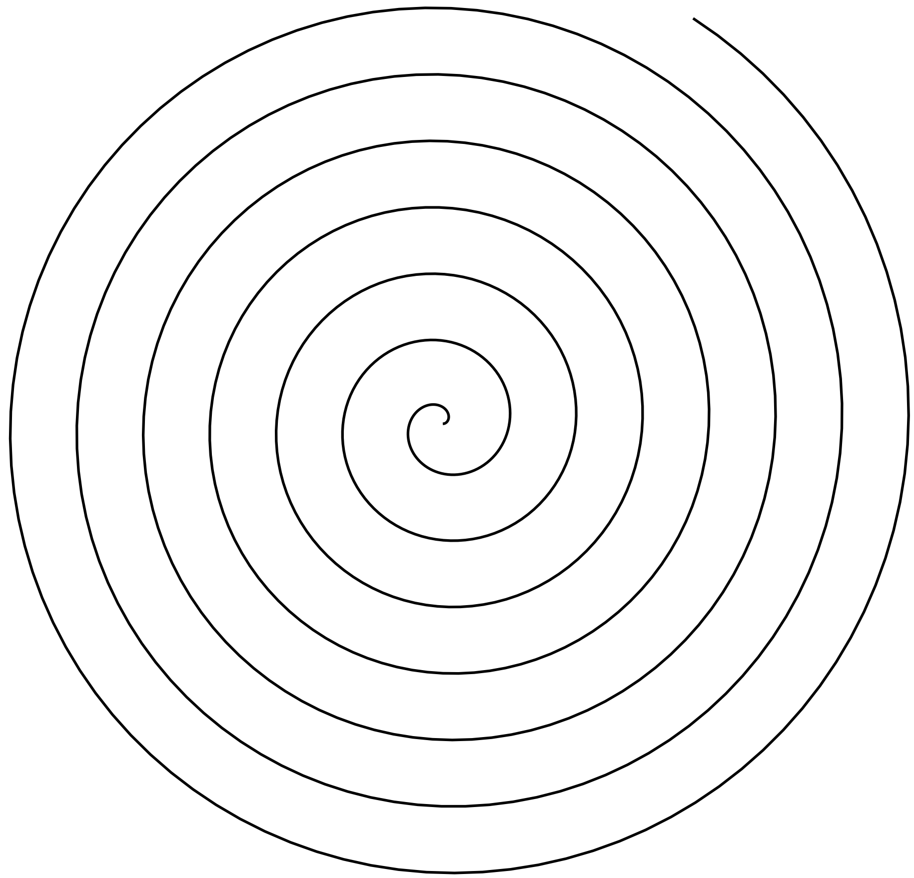


Strip tree for circle



Strip tree for spiral



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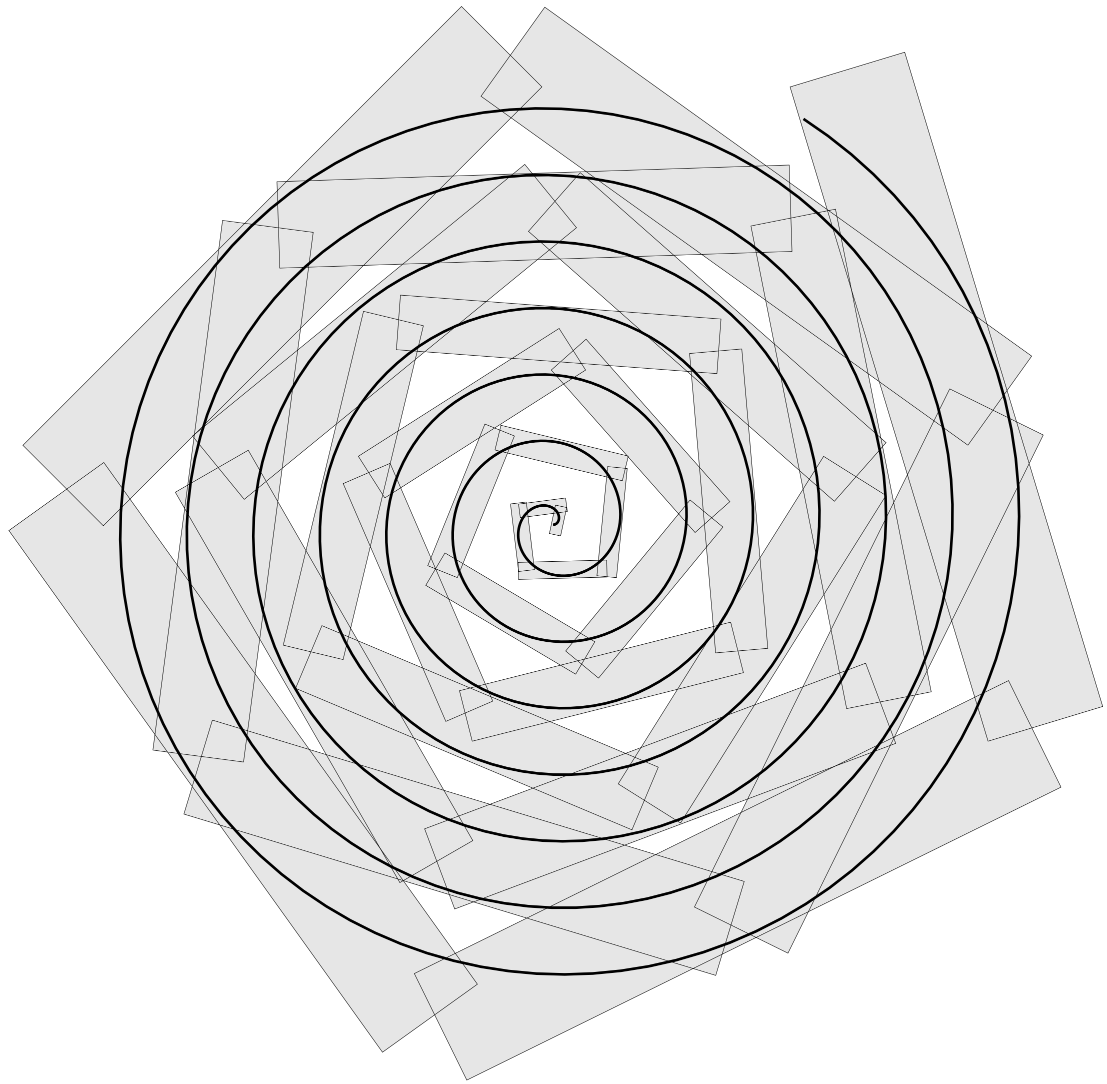

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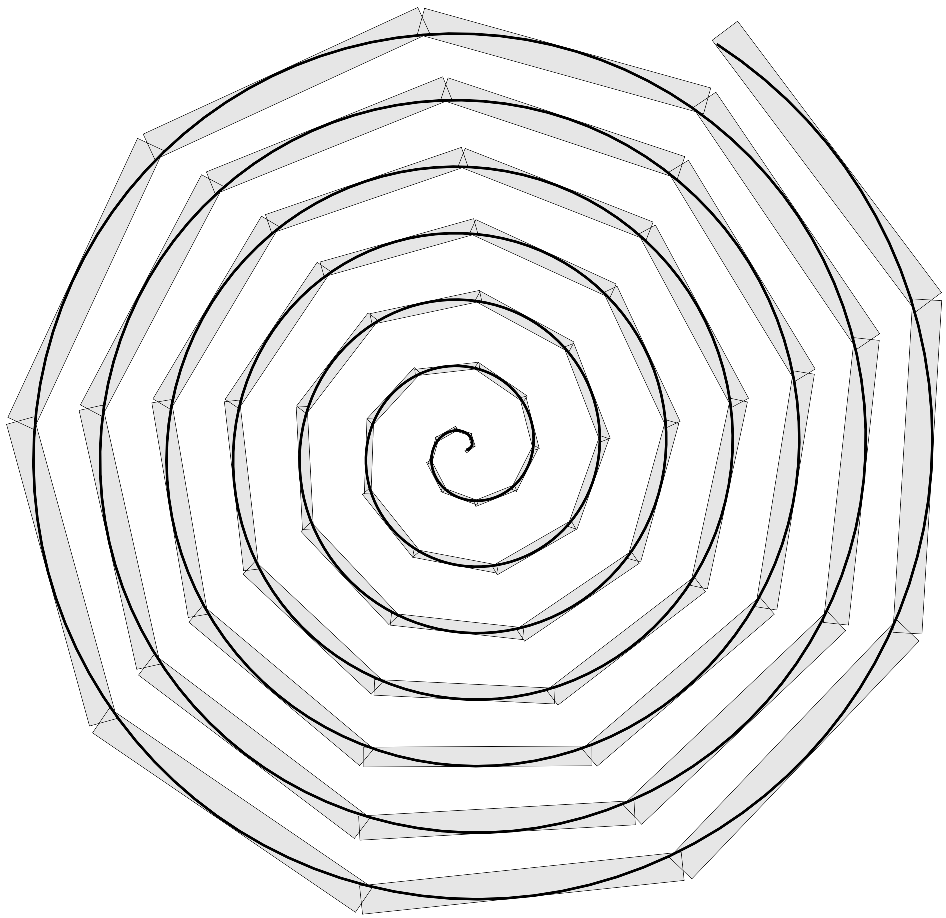

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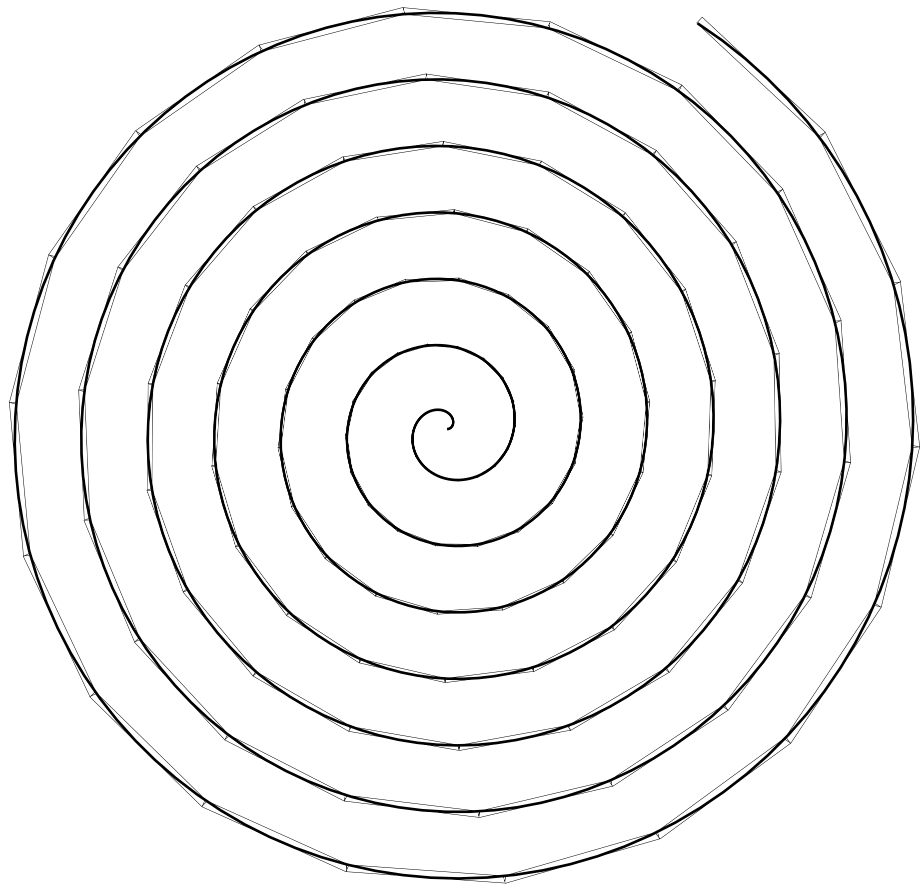

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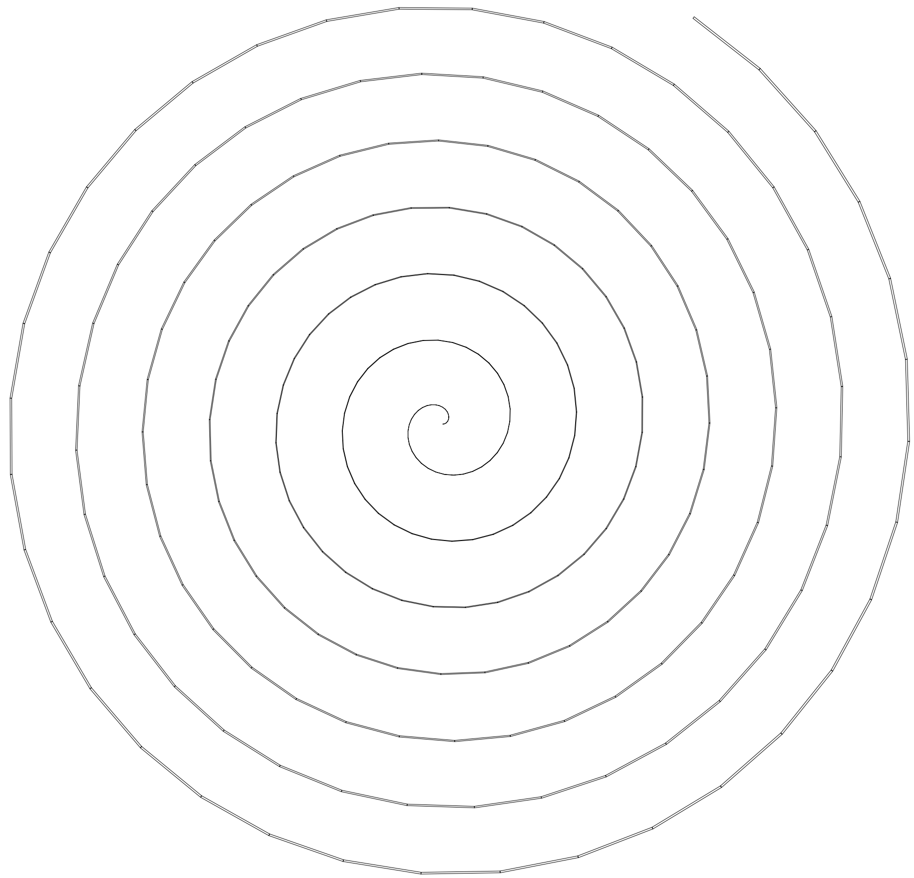
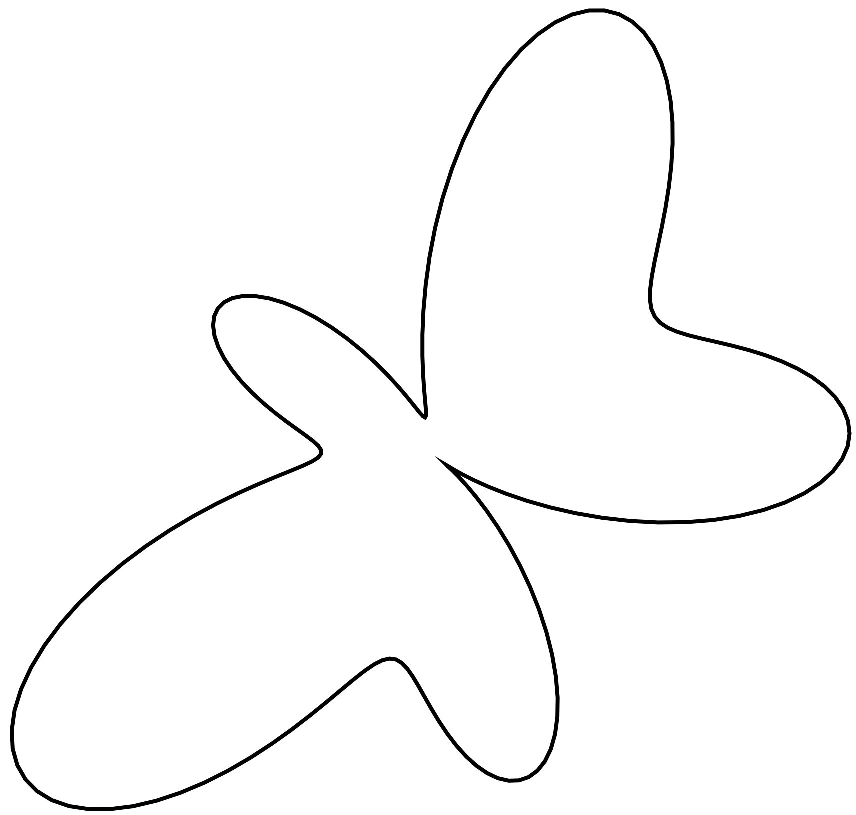


Strip tree for butterfly



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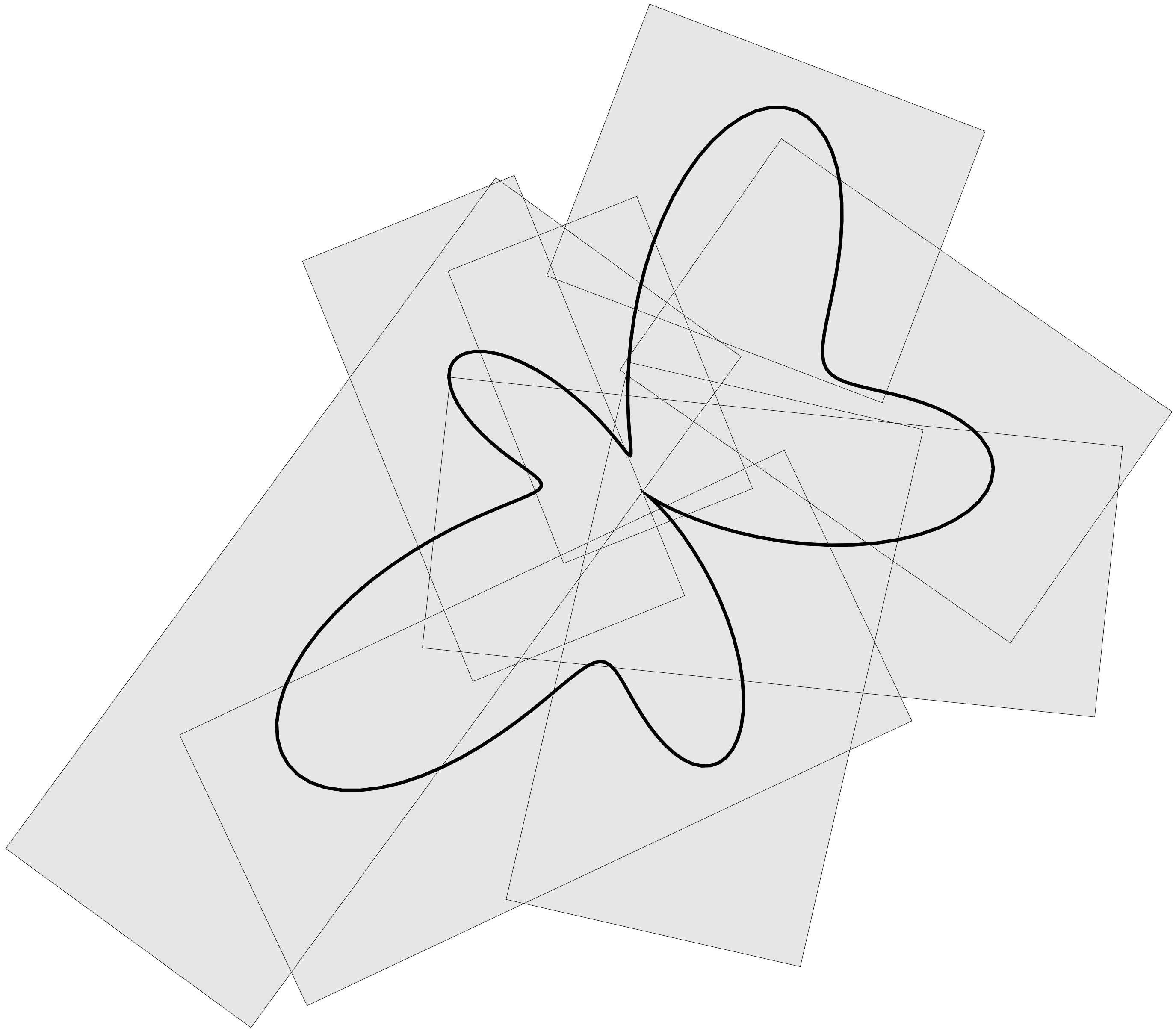

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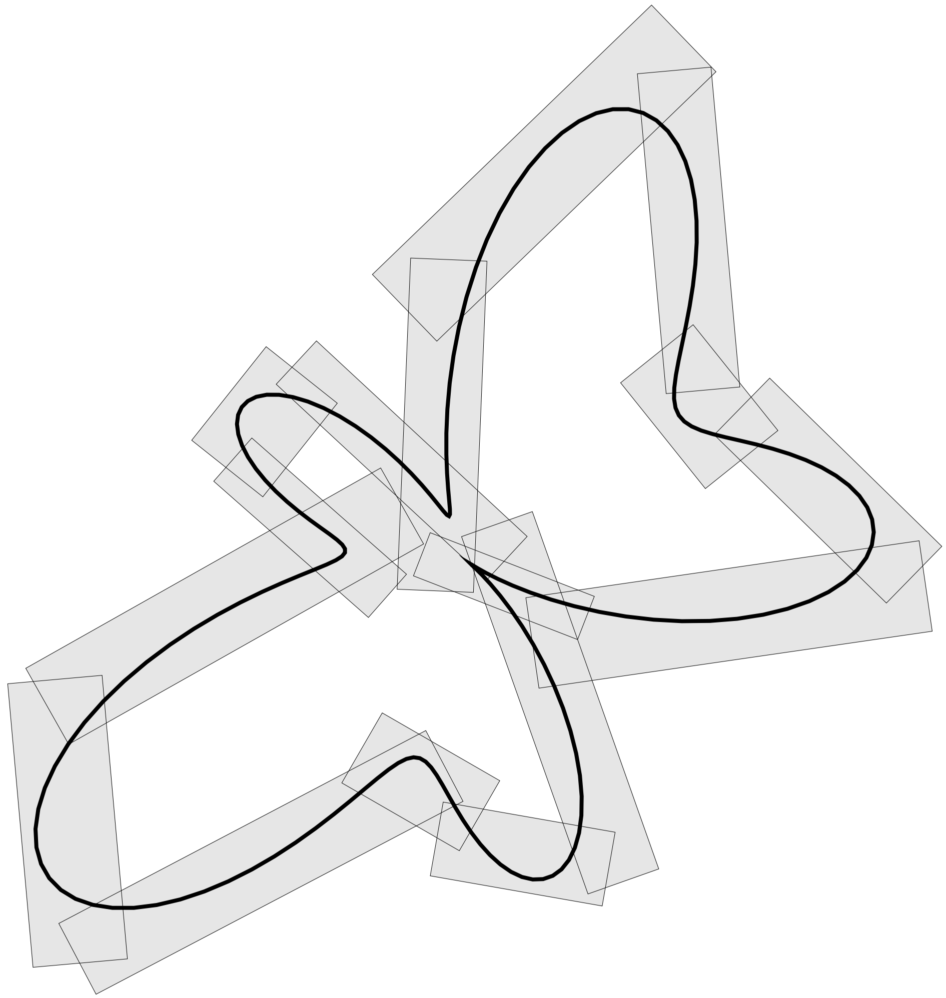

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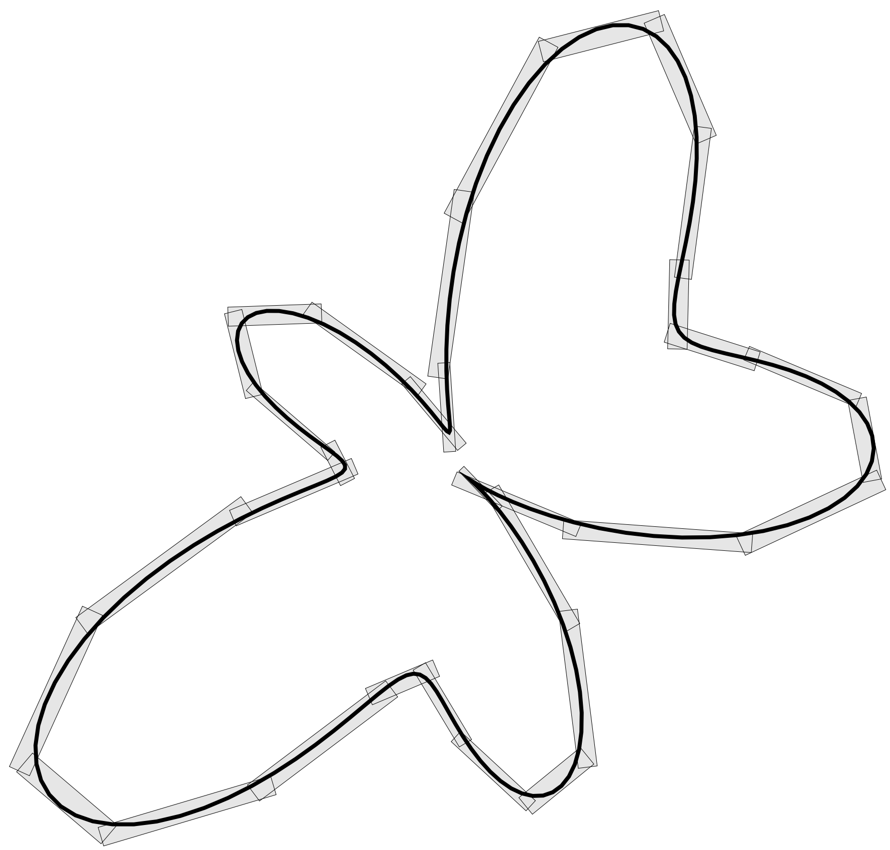

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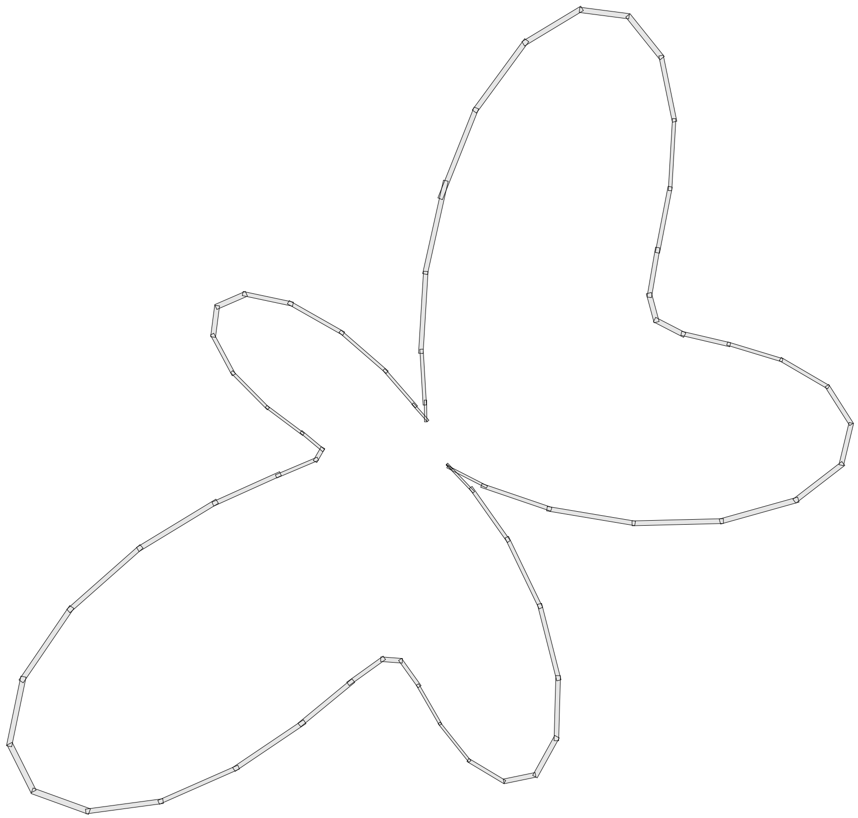
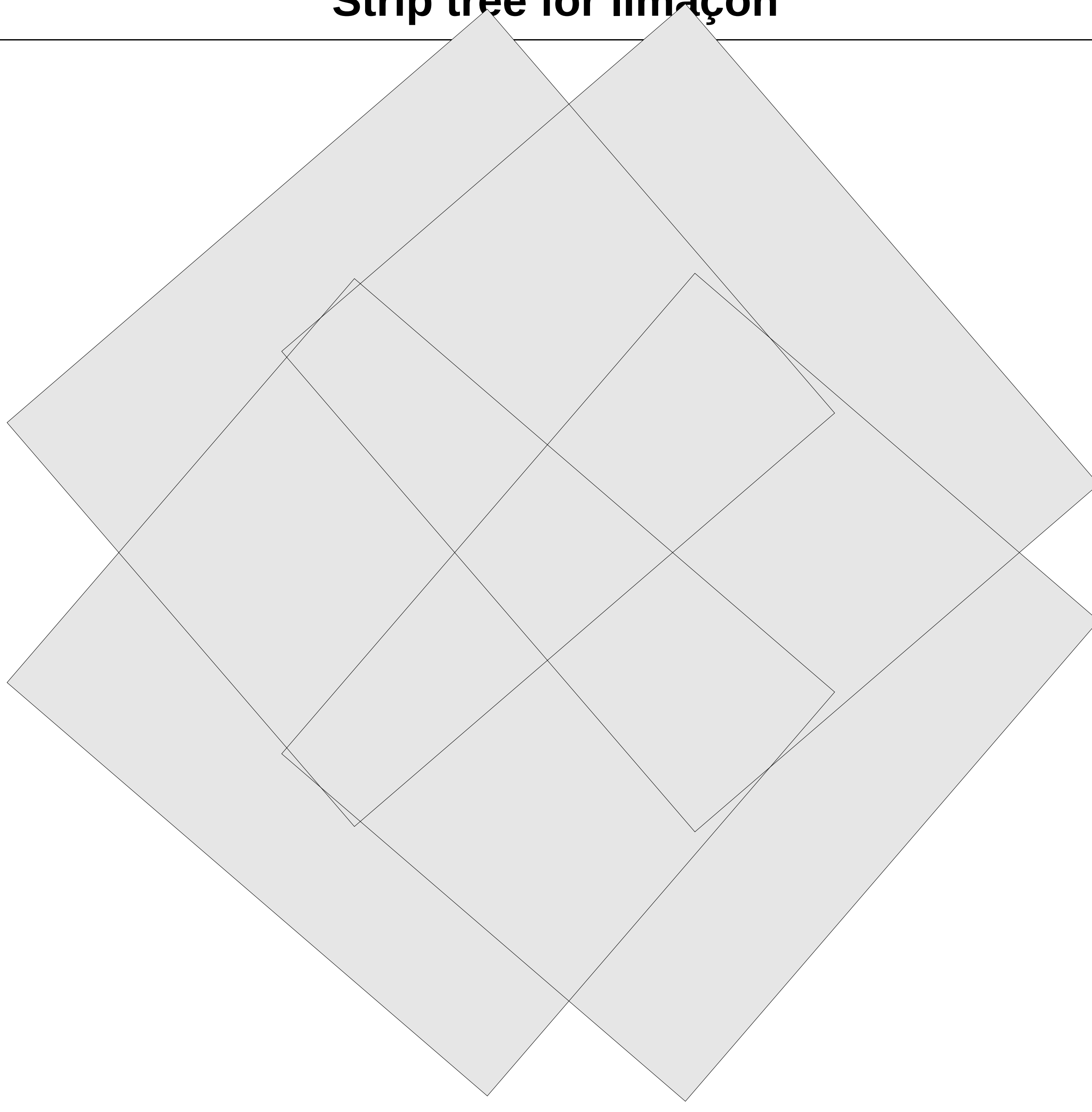


Strip tree for limaçon



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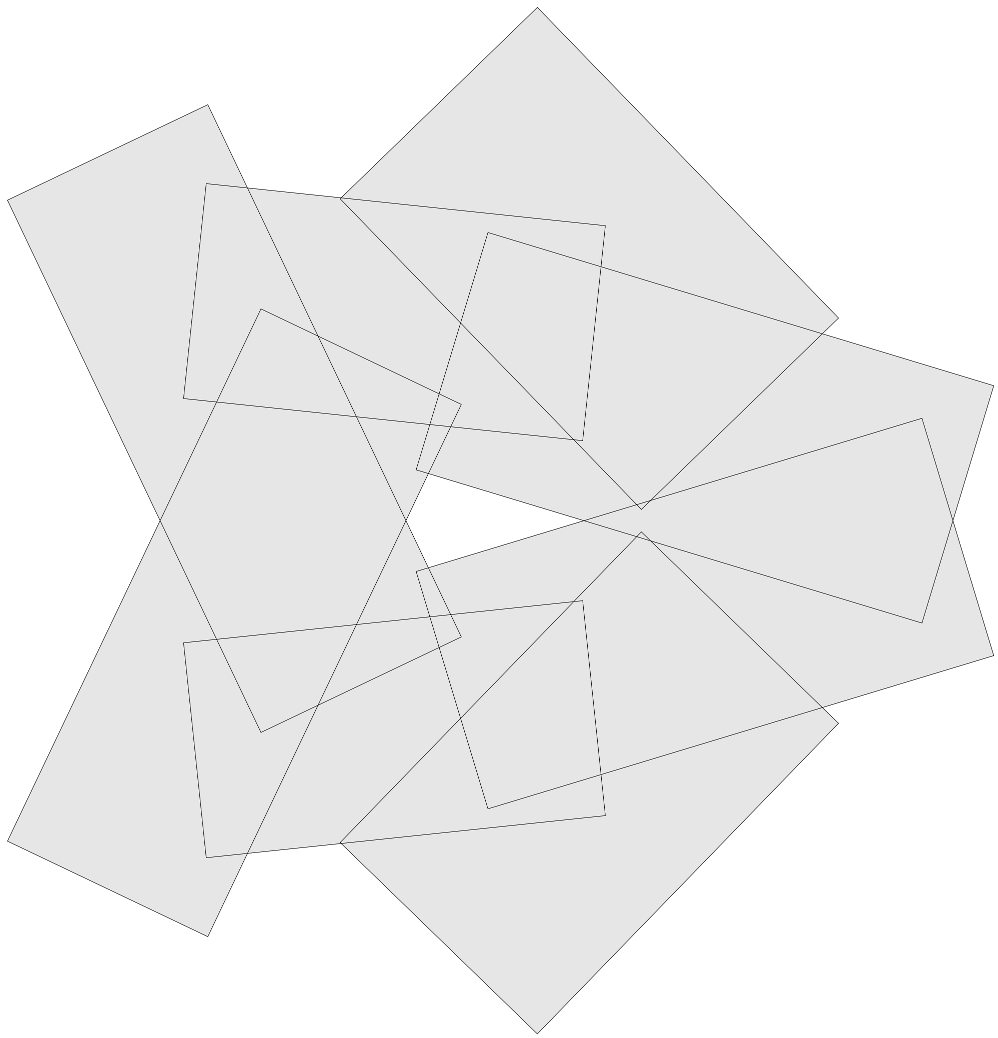

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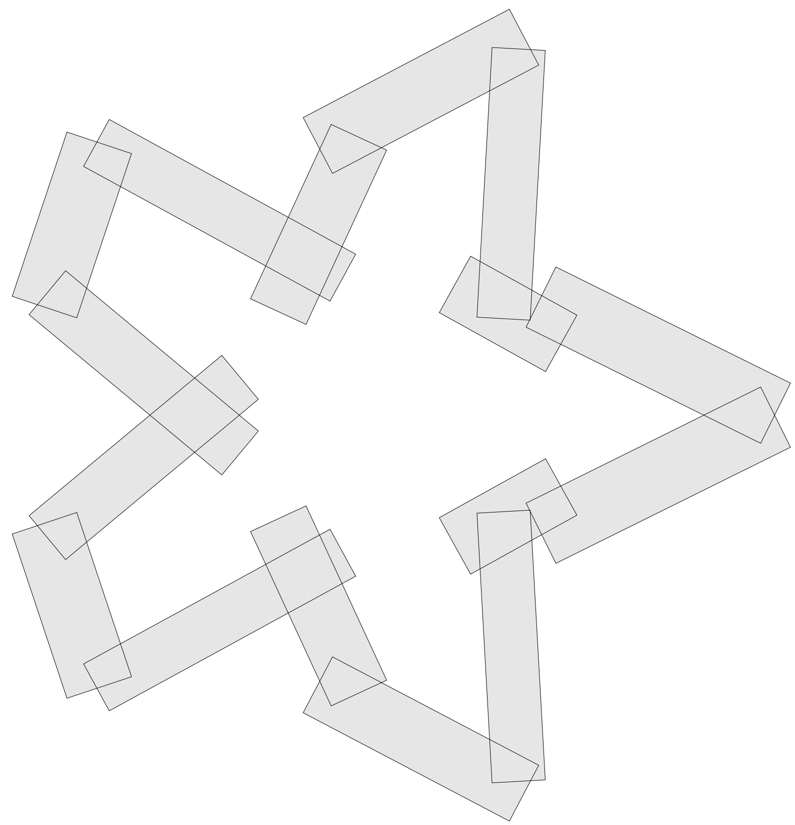

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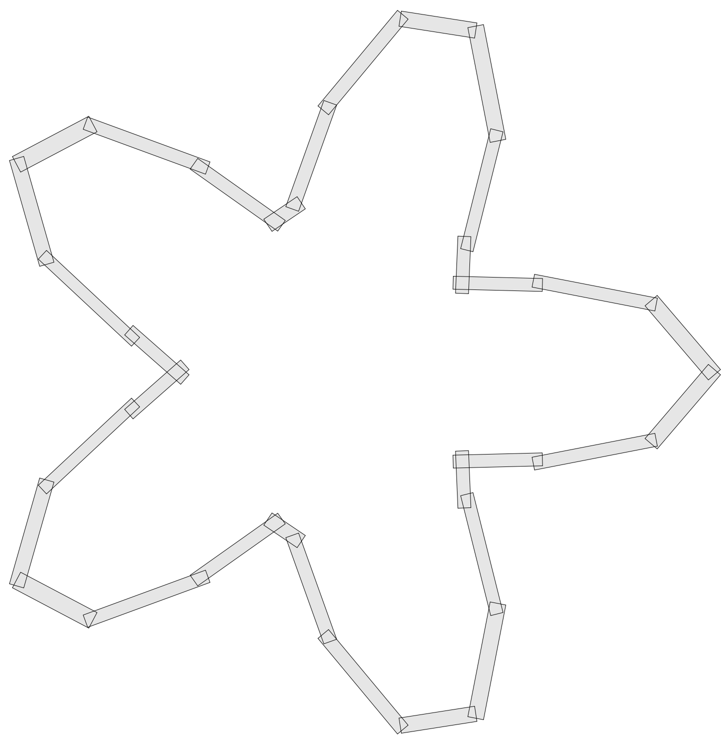

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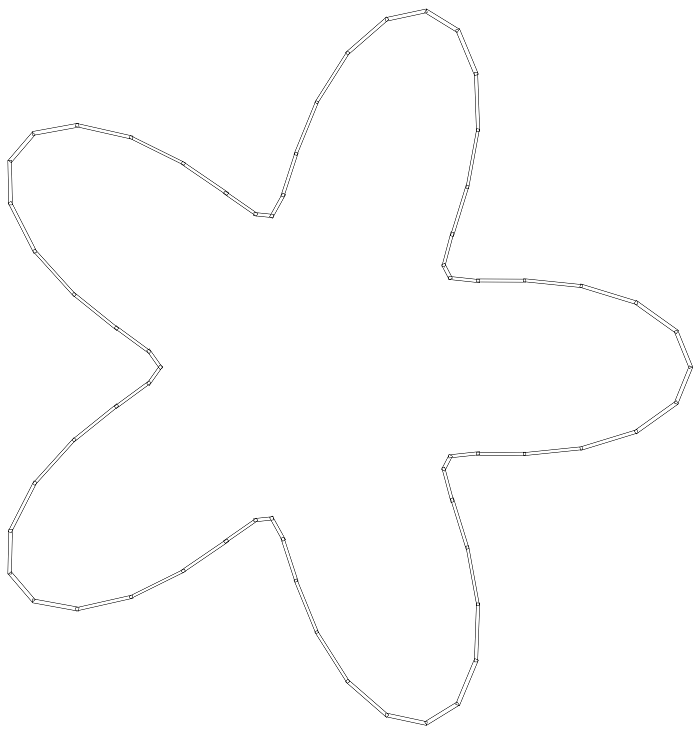
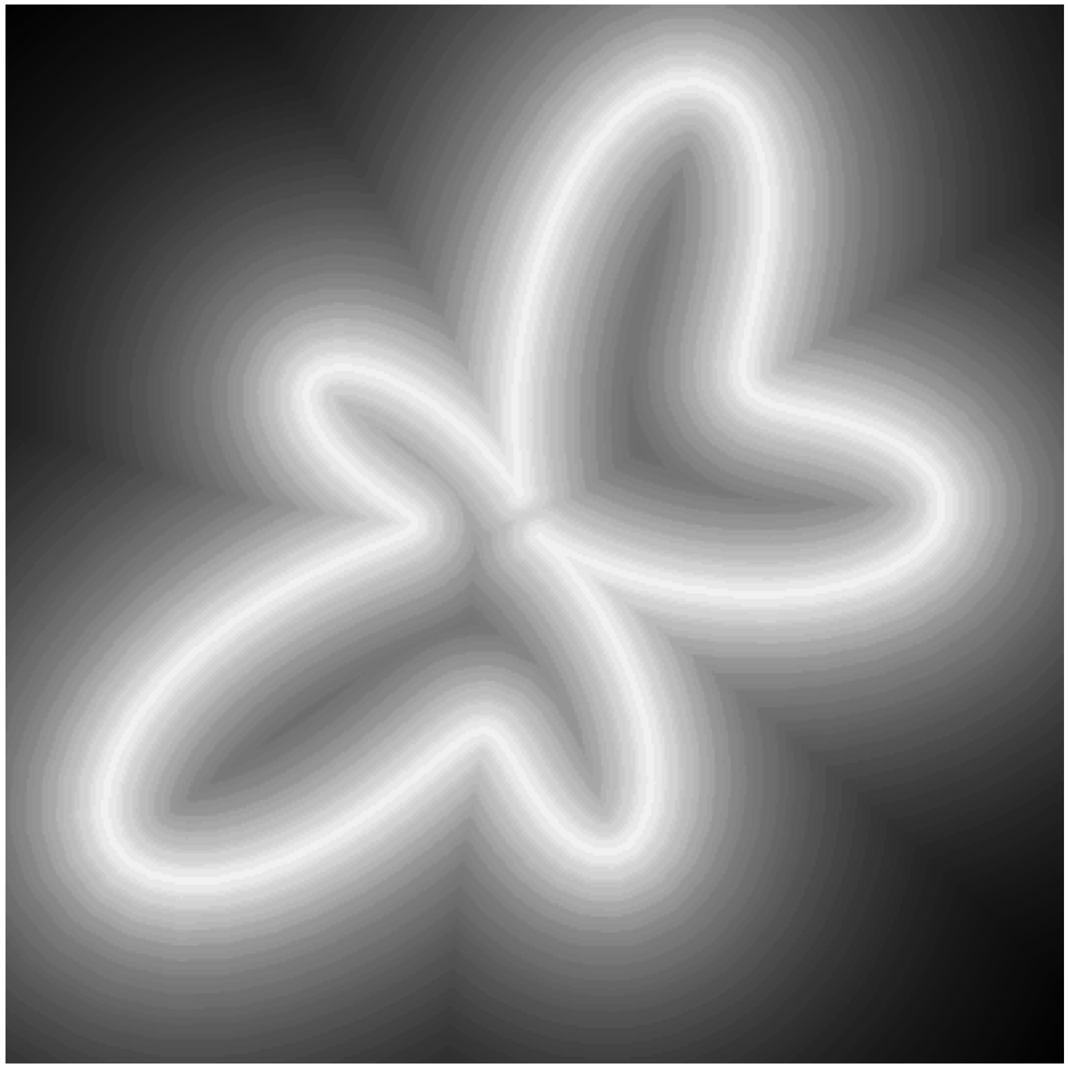
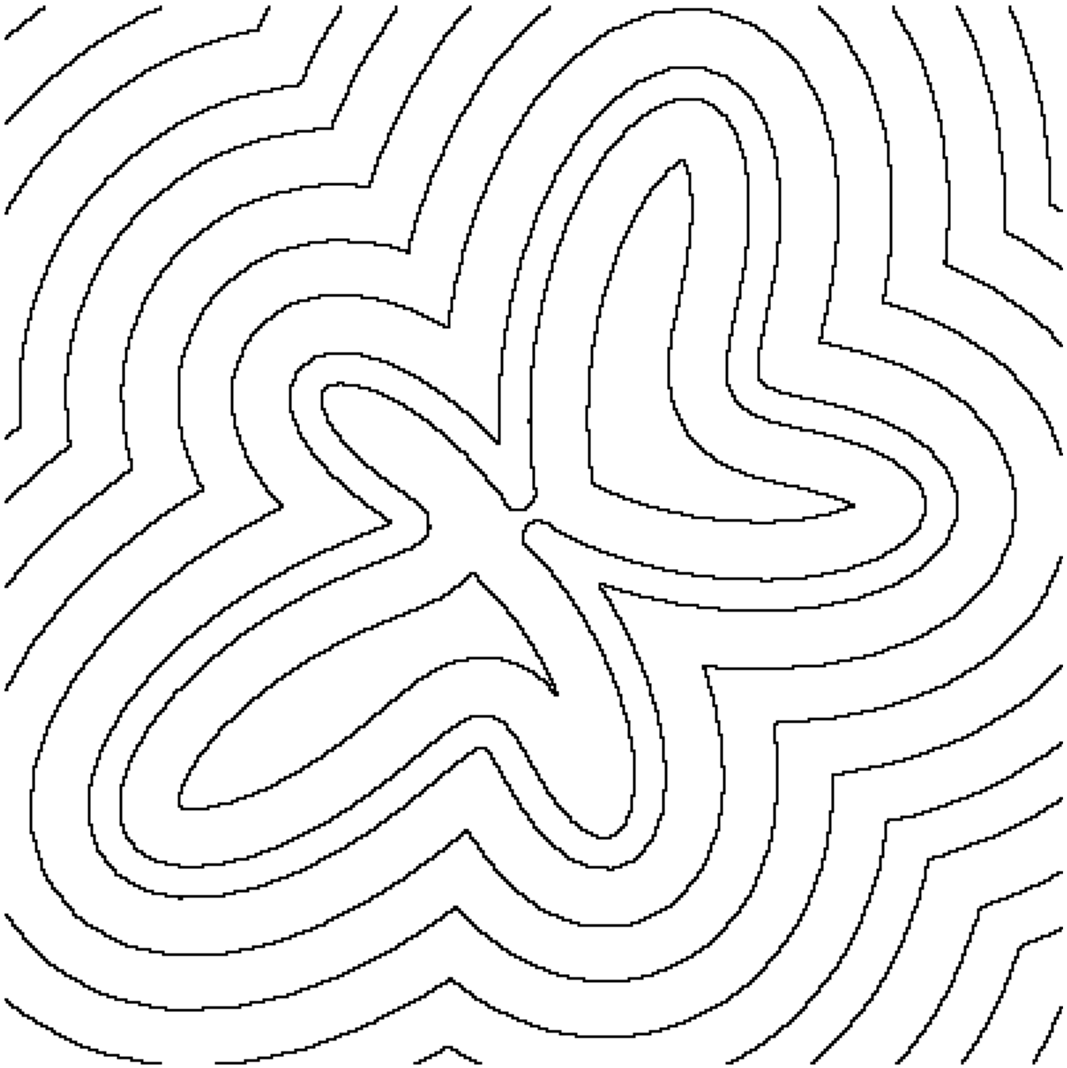


Distance field for butterfly



Offsets for butterfly



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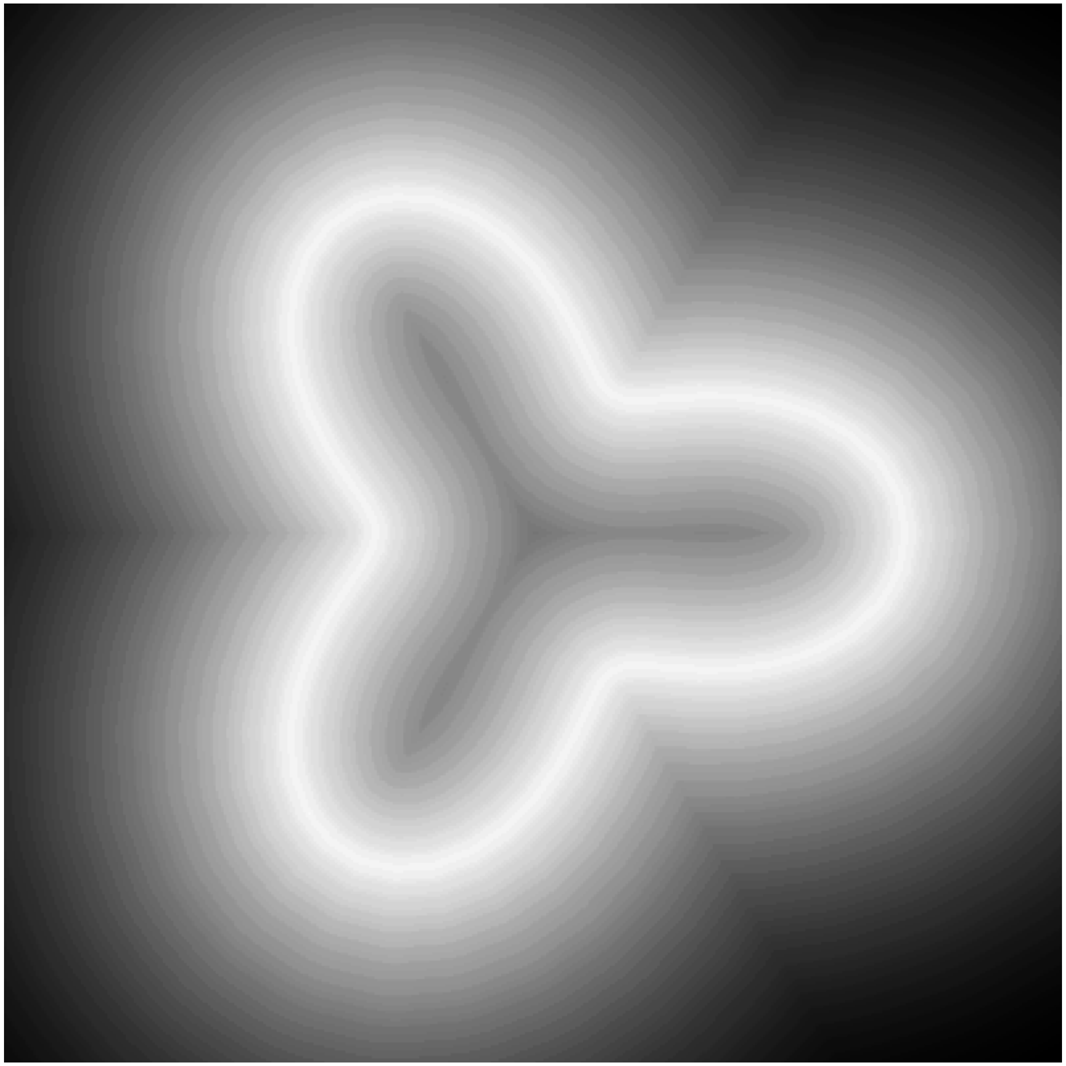

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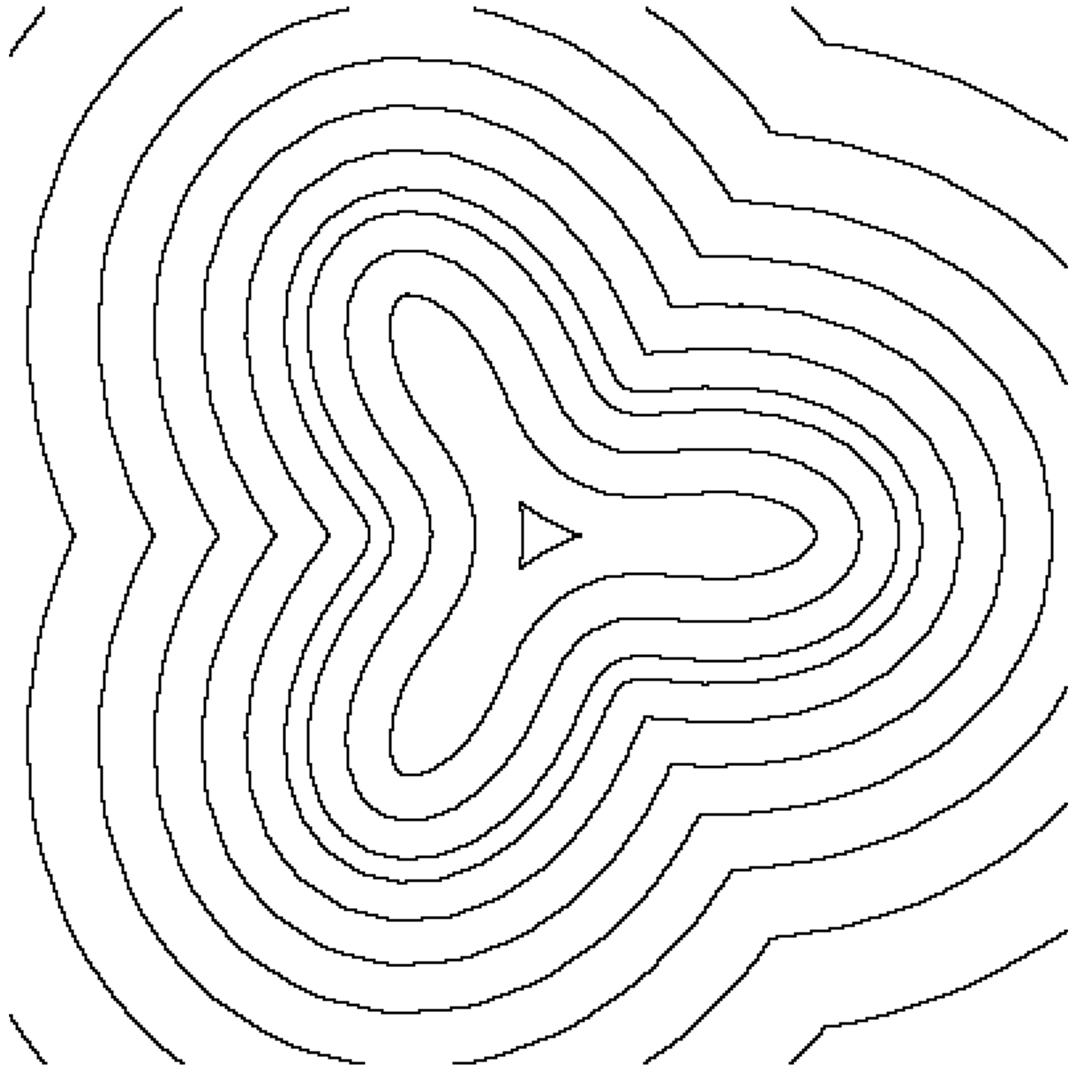

Conclusion

- Strip trees for general parametric curves
  - ◇ non-aligned bounding rectangles from zonotopes given by affine arithmetic

- Implicit approximation of parametric curves via distance fields

- Future work: Surfaces
  - ◇ non-aligned rectangular boxes from 3D zonotopes (how?)
  - ◇ domain decomposition (how?)
    - · 4-8 meshes seem to be convenient for affine arithmetic