A vertex-centric representation for adaptive diamond-kite meshes

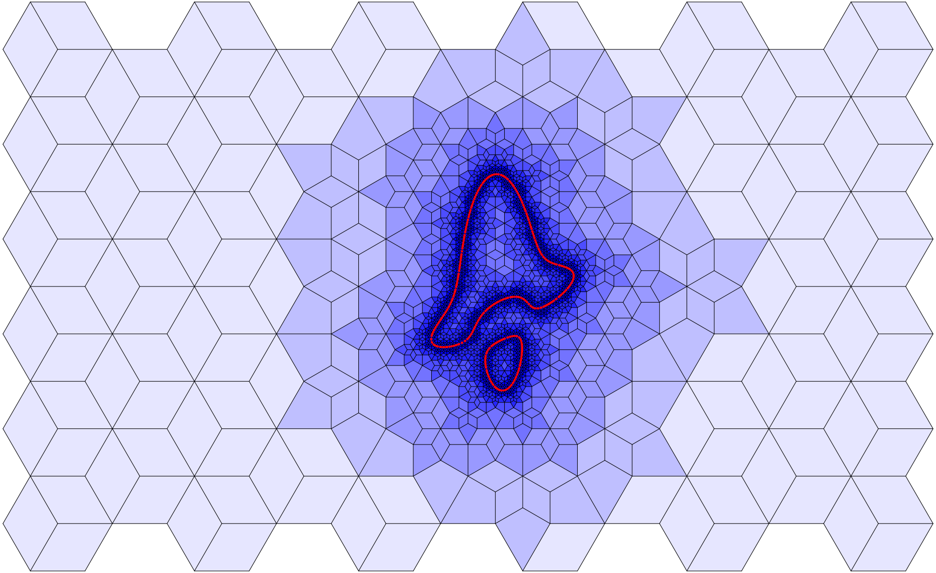
Luiz Henrique de Figueiredo

impa

Visgraf Vision and Graphics Laboratory

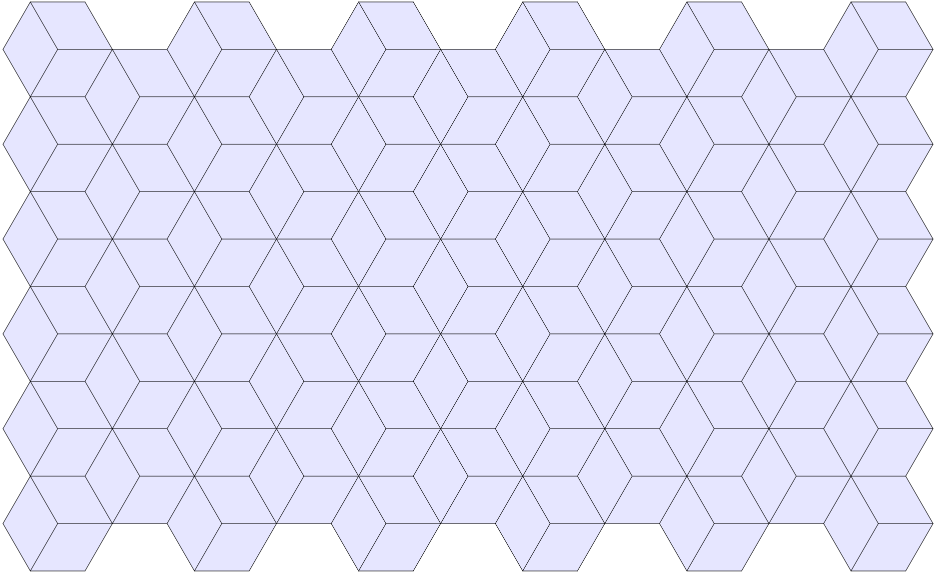
Adaptive diamond-kite meshes

Eppstein (2014)



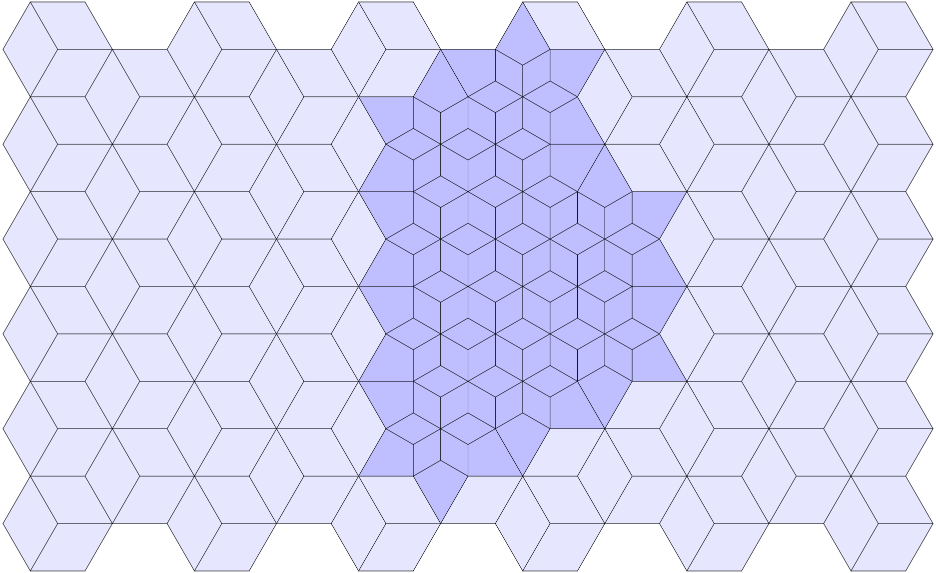
Adaptive diamond-kite meshes

Eppstein (2014)



Adaptive diamond-kite meshes

Eppstein (2014)



Adaptive diamond-kite meshes

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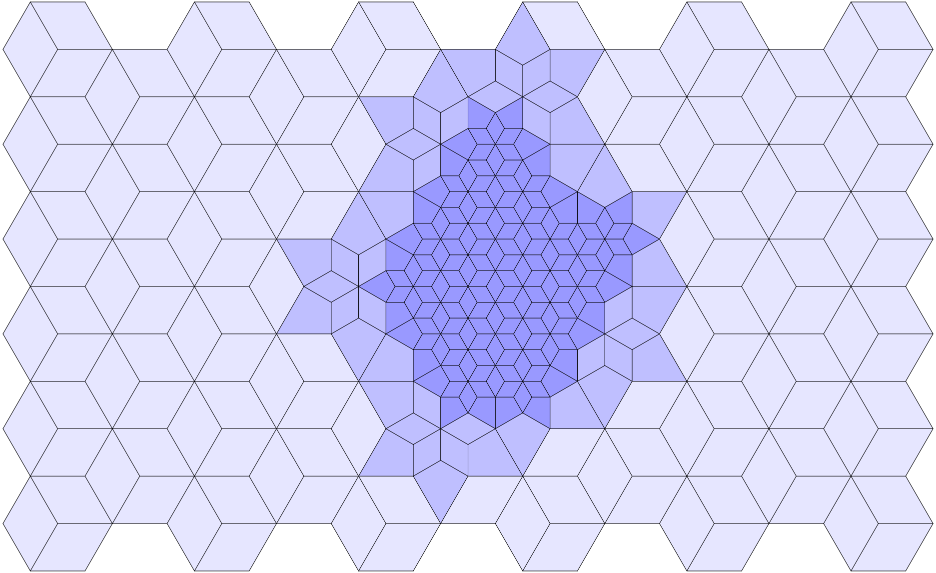
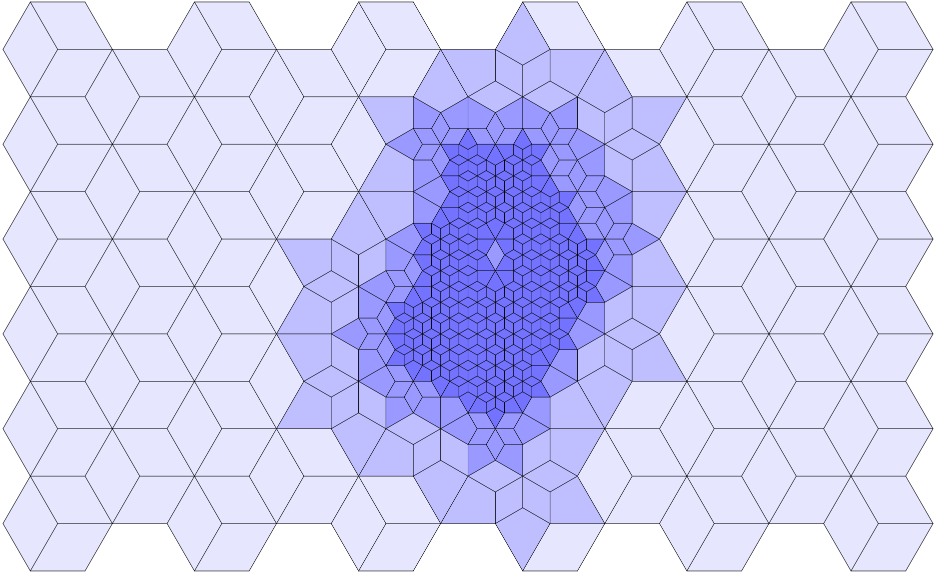


Adaptive diamond-kite meshes

Eppstein (2014)



Adaptive diamond-kite meshes

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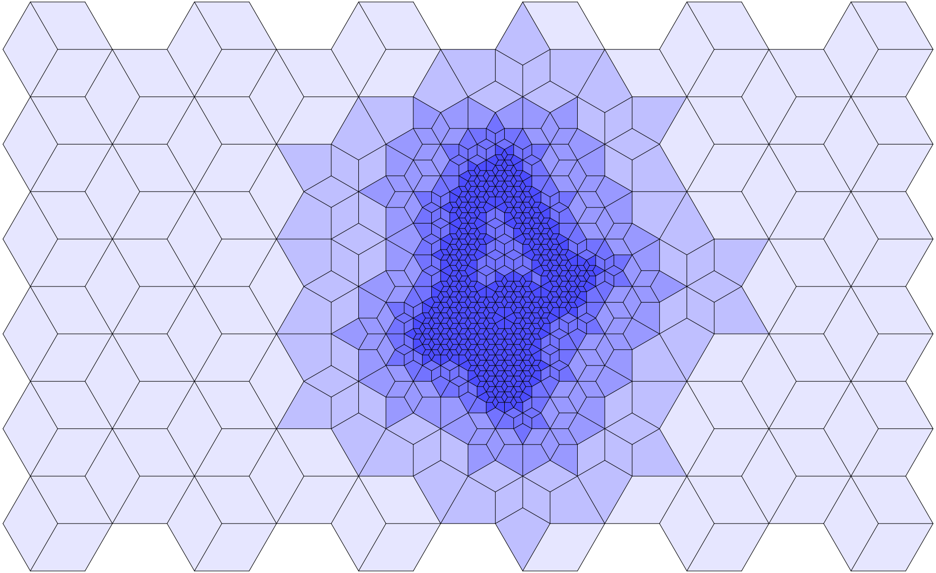
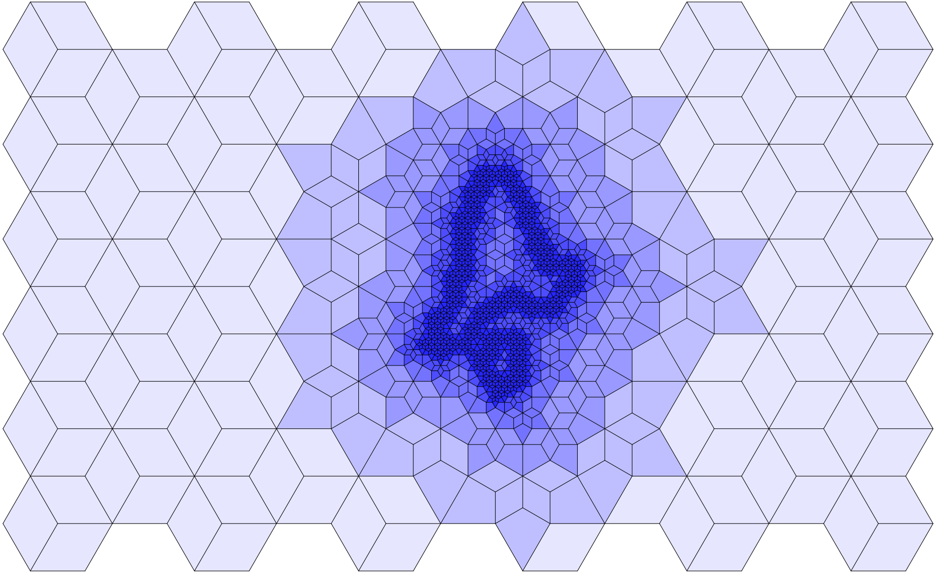


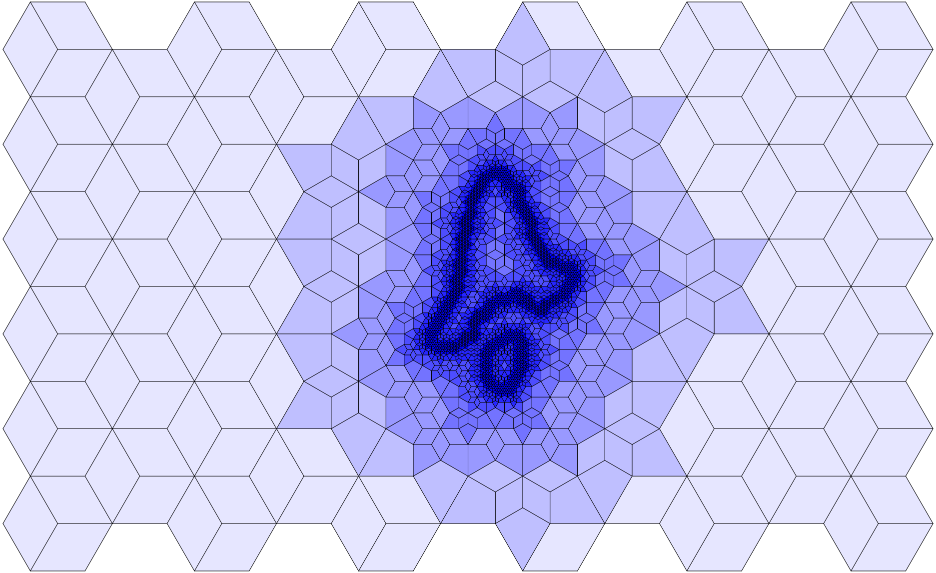
Adaptive diamond-kite meshes

Eppstein (2014)



Adaptive diamond-kite meshes

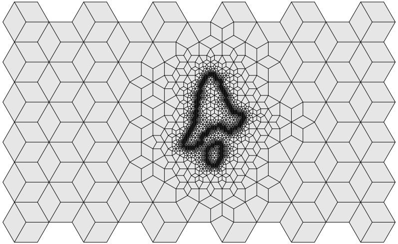
Eppstein (2014)



Adaptive diamond-kite meshes

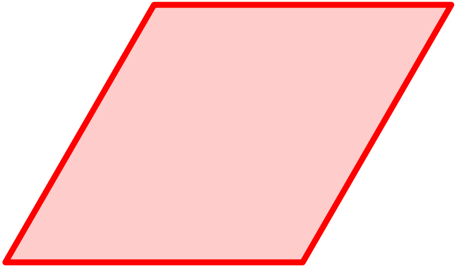
Eppstein (2014)

- quadrilateral meshes
- refined recursively using local subdivision operations
- faces of bounded aspect ratio
- invariant under Laplacian smoothing

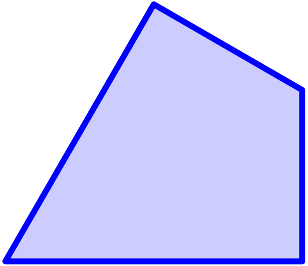


adaptive diamond-kite meshes
concepts

Adaptive diamond-kite meshes – faces

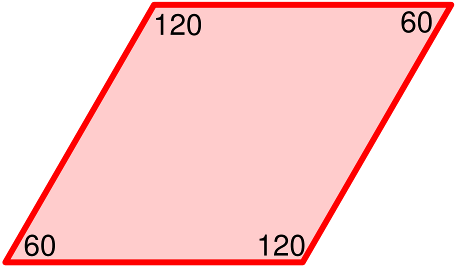


diamond

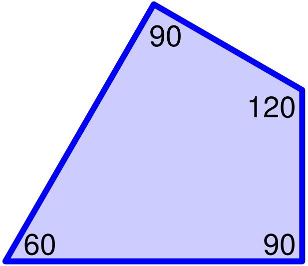


kite

Adaptive diamond-kite meshes – faces

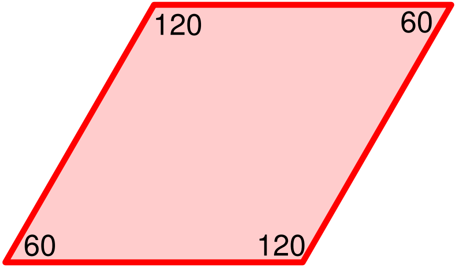


diamond



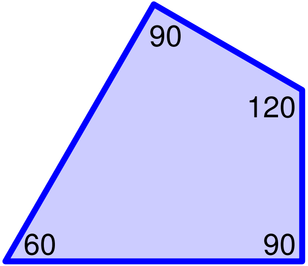
kite

Adaptive diamond-kite meshes – faces



diamond

sides L

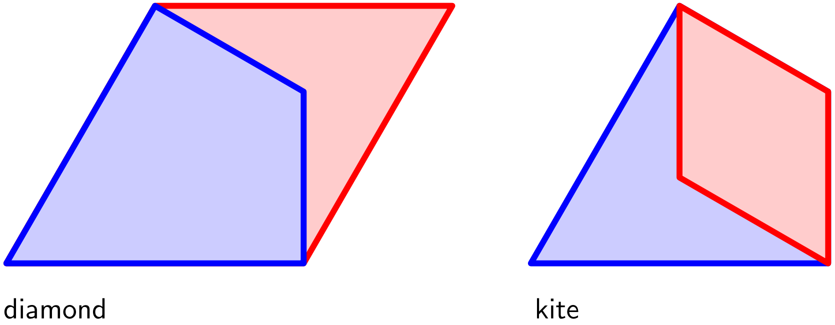


kite

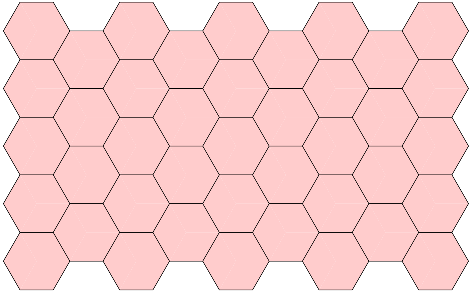
sides L and ρL

$\rho = \tan(30^\circ) = \frac{1}{\sqrt{3}} \approx 0.577$

Adaptive diamond-kite meshes – faces

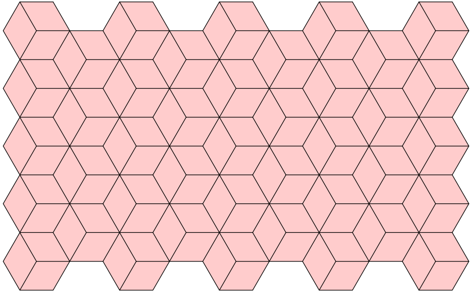


Adaptive diamond-kite meshes – base mesh



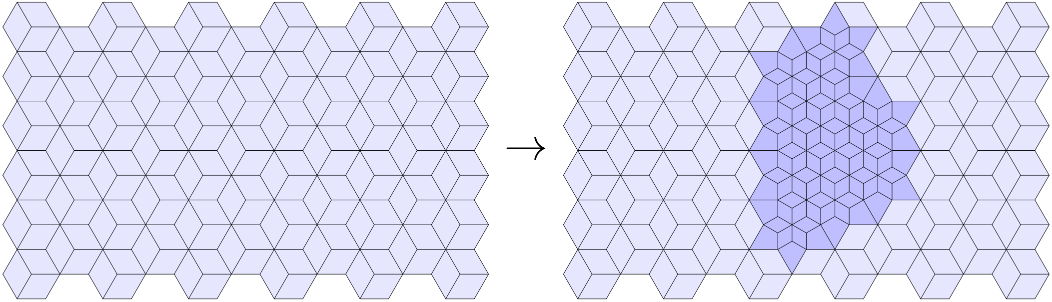
hexagonal tiling

$\rightarrow$



base mesh = finite rhombille tiling

Adaptive diamond-kite meshes – refinement



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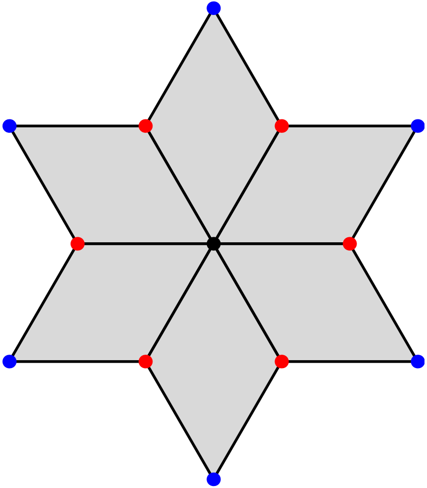
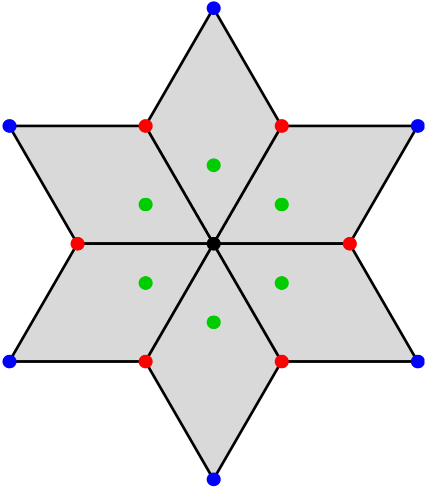


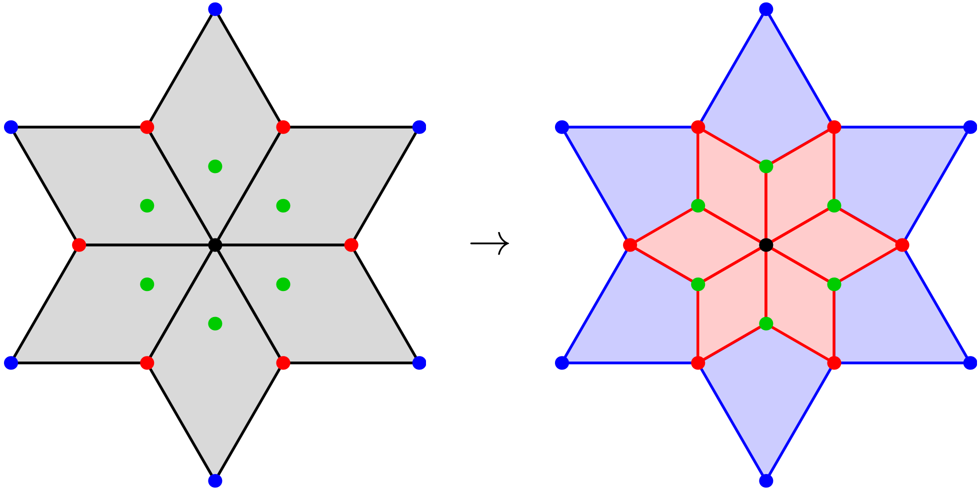
vertex of degree 6

Adaptive diamond-kite meshes – refinement



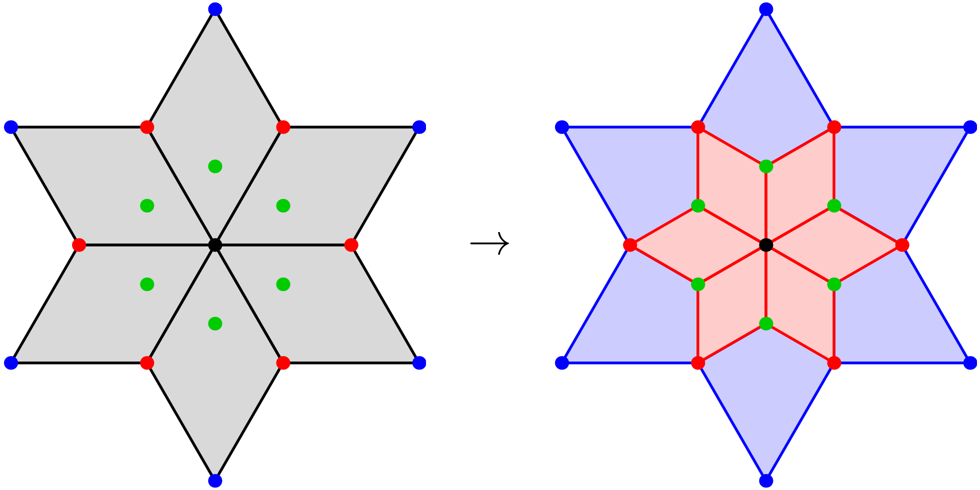
barycenters

Adaptive diamond-kite meshes – refinement



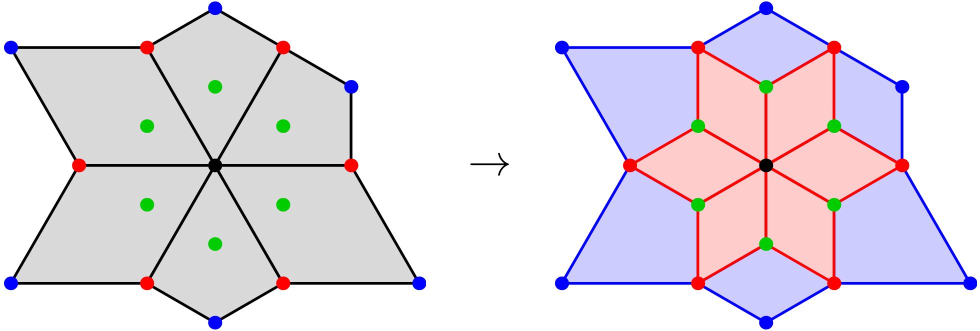
remeshing

Adaptive diamond-kite meshes – refinement



new central edges rotated 30° and scaled by $\rho \approx 0.577$

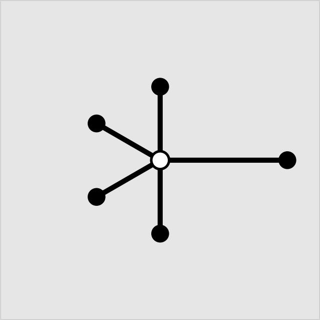
Adaptive diamond-kite meshes – refinement



original faces can be any combination of diamonds and kites

Adaptive diamond-kite meshes – stars

star of a vertex = circular sequence of vertices adjacent to it



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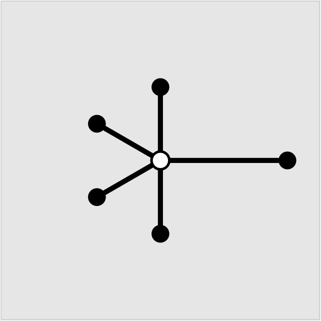

distribution of angles constrained by integer solutions of
$60x + 90y + 120z = 360$

Adaptive diamond-kite meshes – stars

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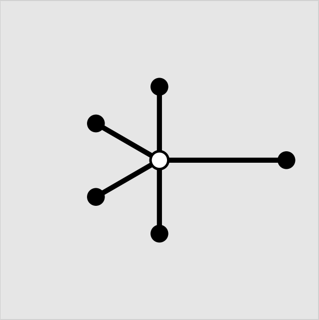

distribution of angles constrained by integer solutions of $60x + 90y + 120z = 360$

x	y	z	degree	arrangements of angles		
0	0	3	3	ccc		
0	4	0	4	bbbb		
1	2	1	4	abbc	abcb	acbb
2	0	2	4	aacc	acac	
3	2	0	5	aaabb	aabab	
4	0	1	5	aaaac		
6	0	0	6	aaaaaa		

a: 60° b: 90° c: 120°

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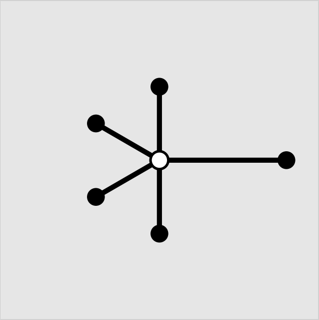

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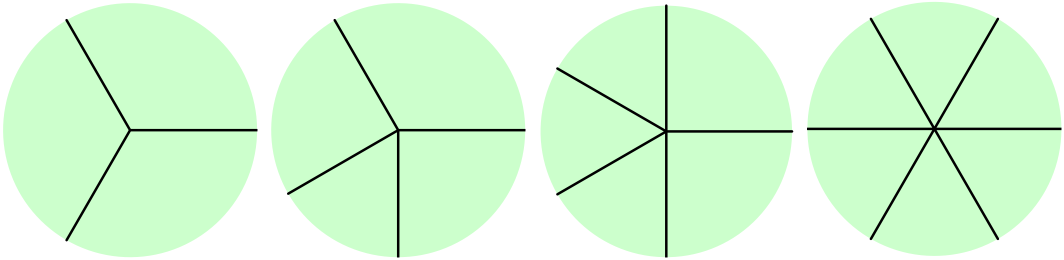
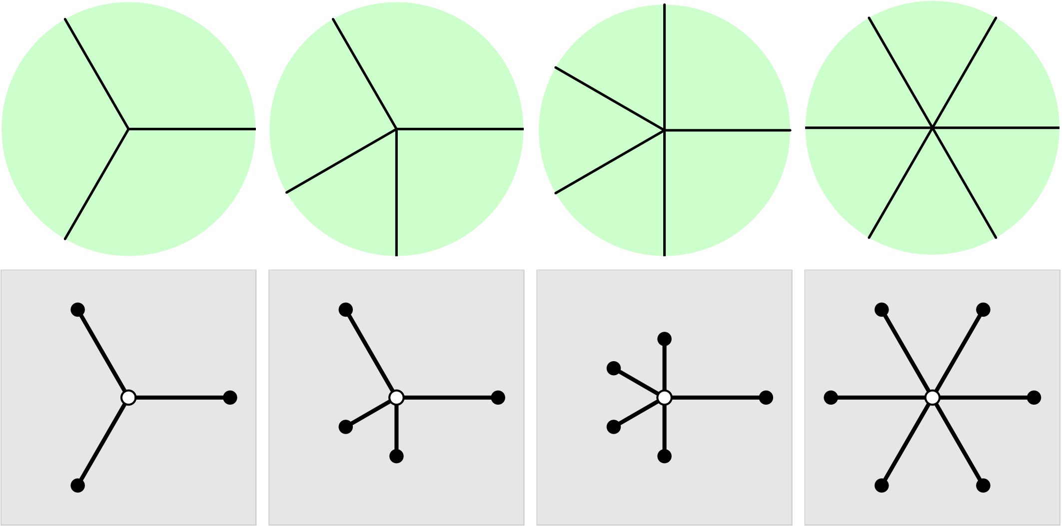
concise vertex-centric representations based on stars

Adaptive diamond-kite meshes – stars



Adaptive diamond-kite meshes – stars



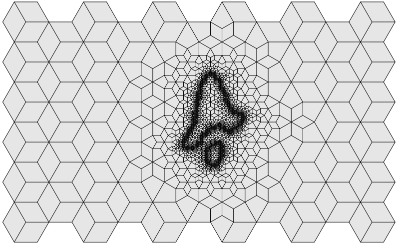
rigidity: only one vertex star for each degree, up to orientation and scale rigid geometry and topology

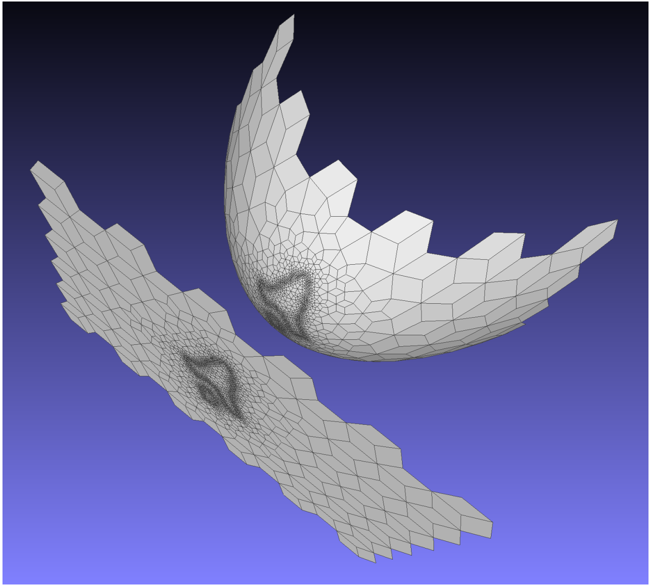
$$\Downarrow$$

concise representation

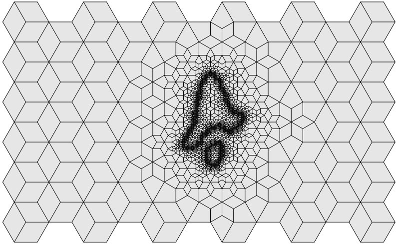
mesh representations

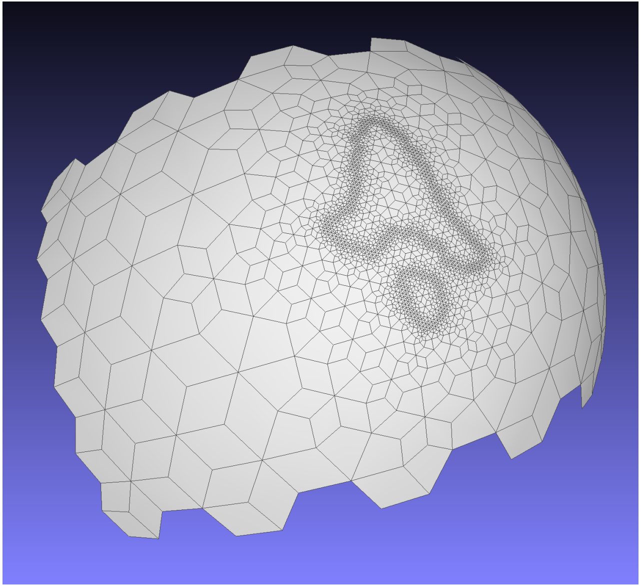
Mesh representations – geometry + topology





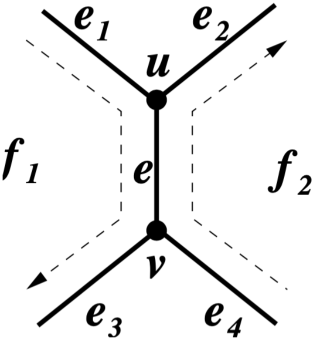
Mesh representations – geometry + topology



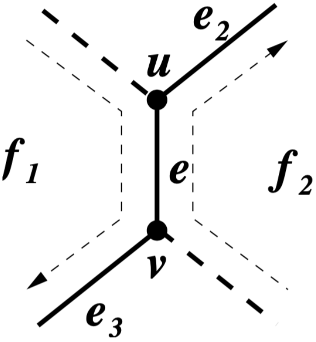


Mesh representations – topological data structures

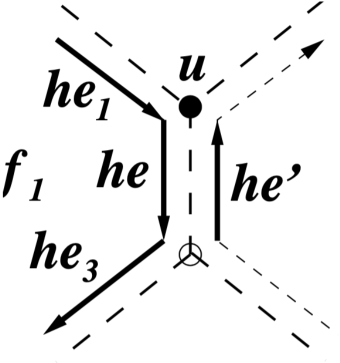
winged edge



dcel



half edge



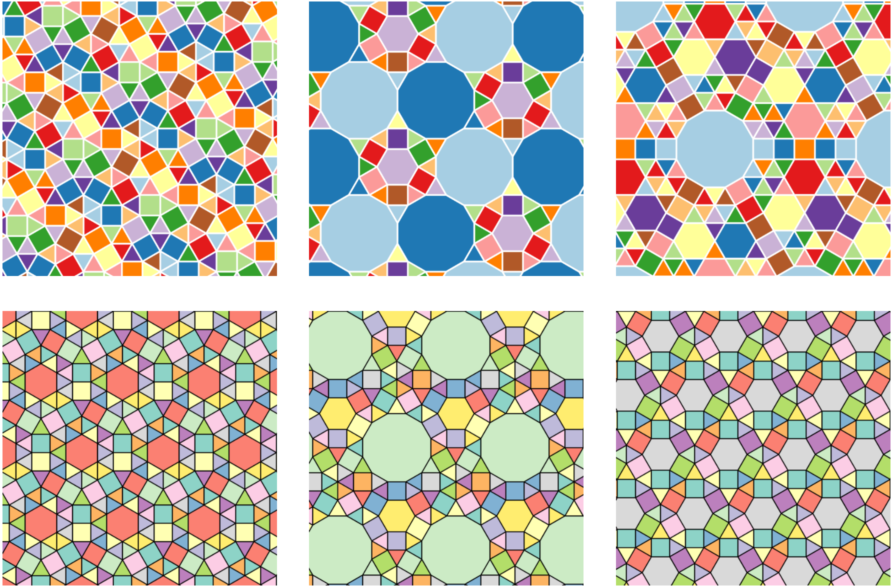
De Floriani–Hui, Shape representations based on simplicial and cell complexes, EG 2007 representation of rigid meshes

=

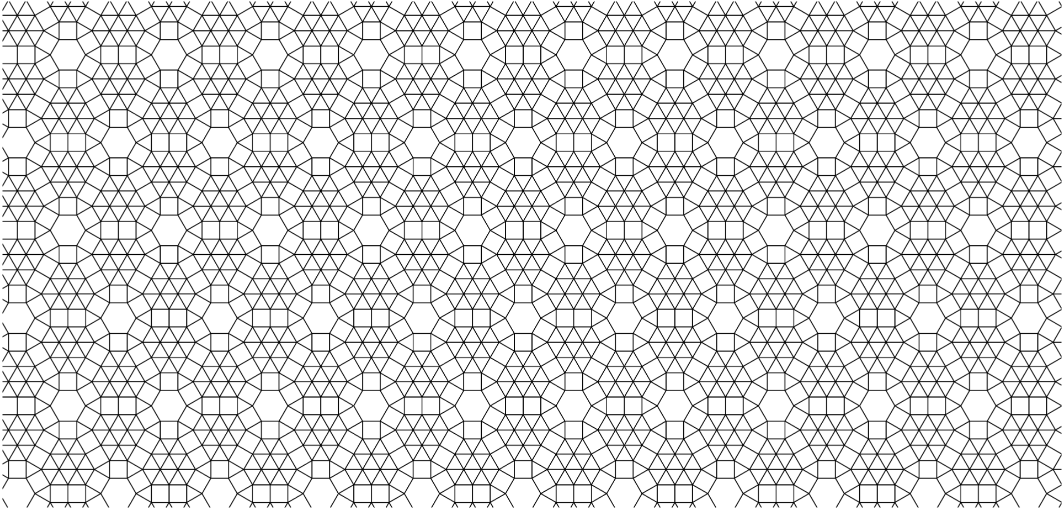
coordinates for vertices

Periodic tilings of the plane by regular polygons

Soto Sánchez (2020)



Examples: Periodic tilings with regular polygons



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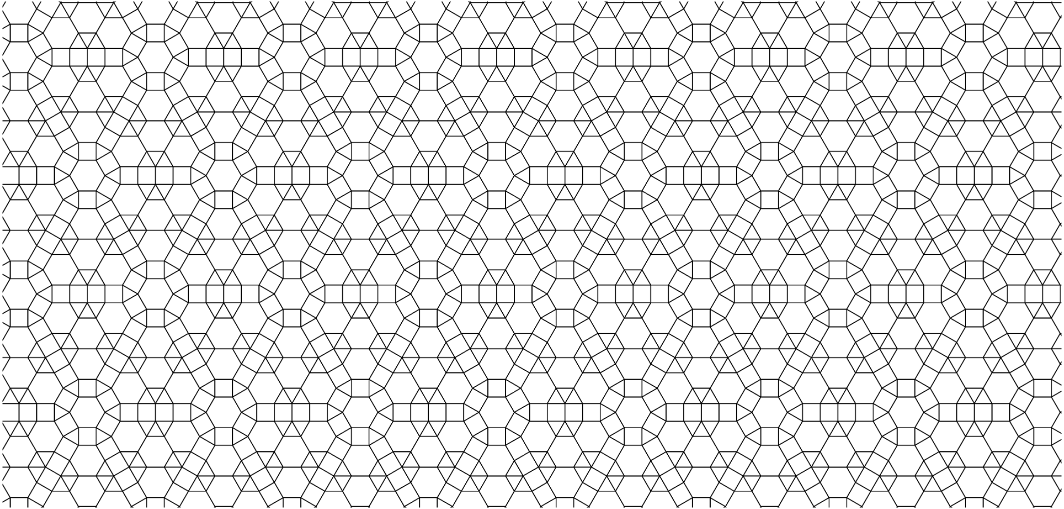

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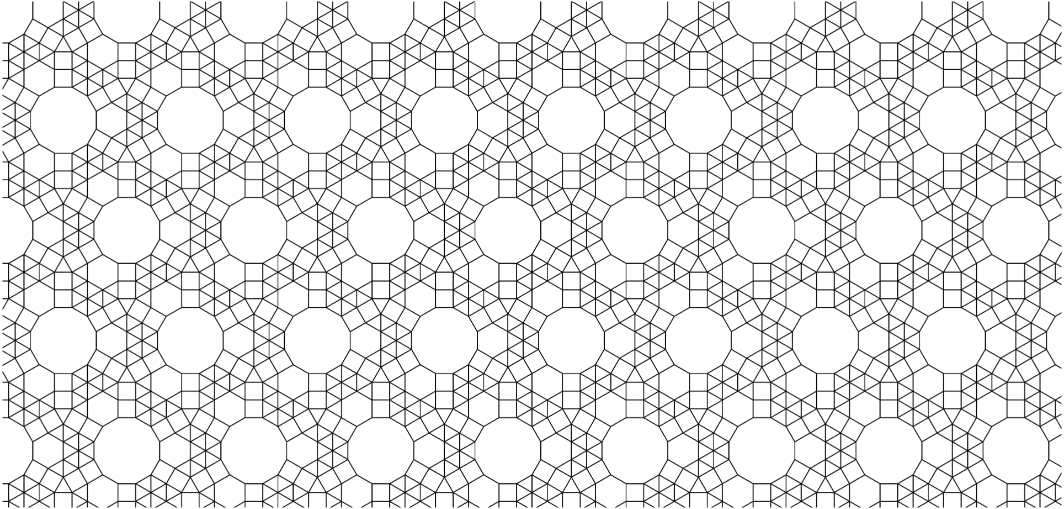

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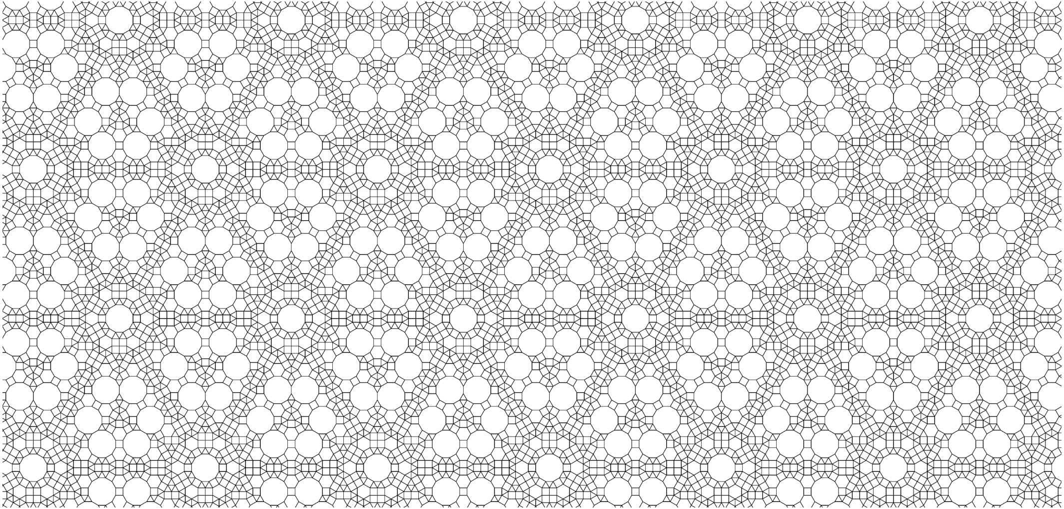
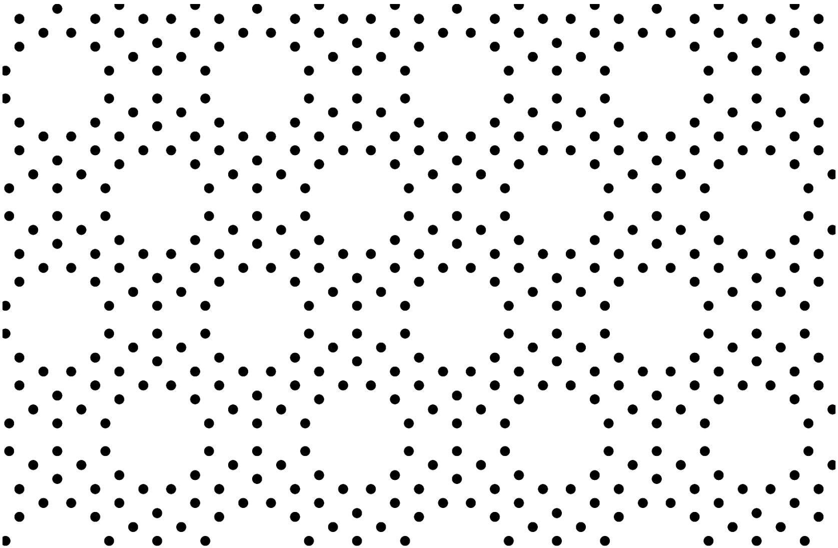
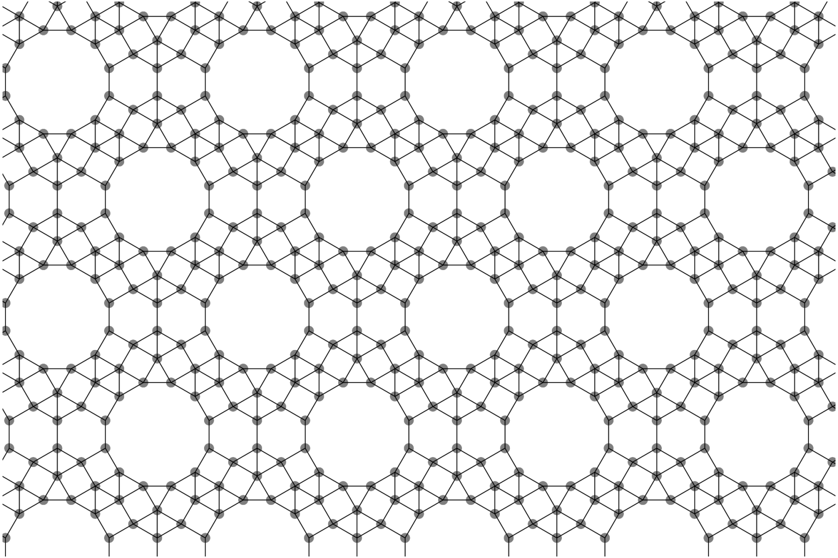


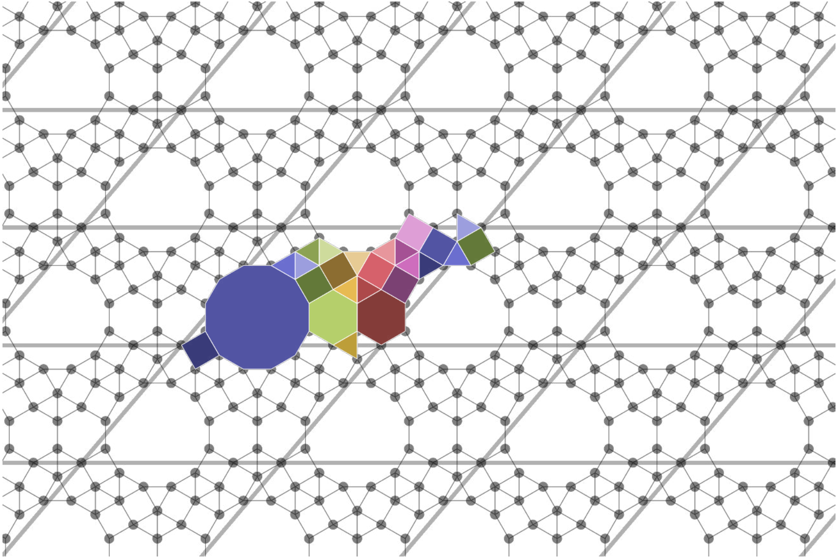
Reconstruct tiling from vertices



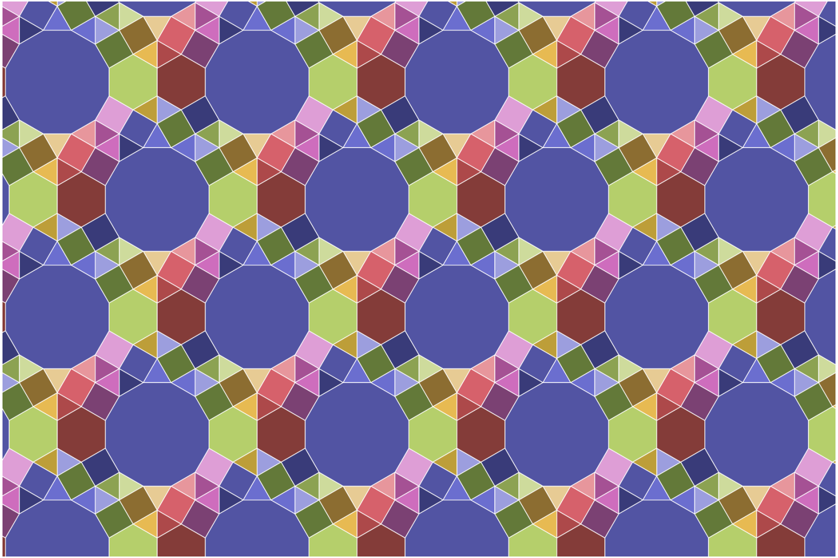
Reconstruct tiling from vertices: edges



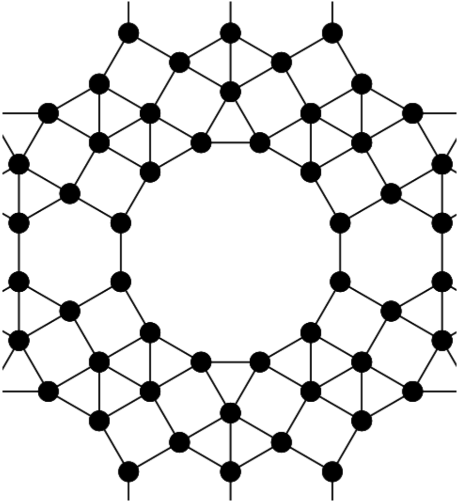
Reconstruct tiling from vertices: patch



Reconstruct tiling from vertices: full tiling



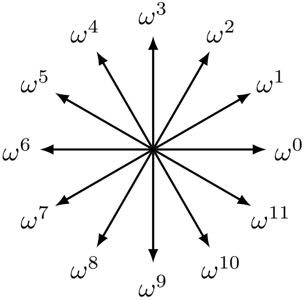
Edges aligned to a few basic directions



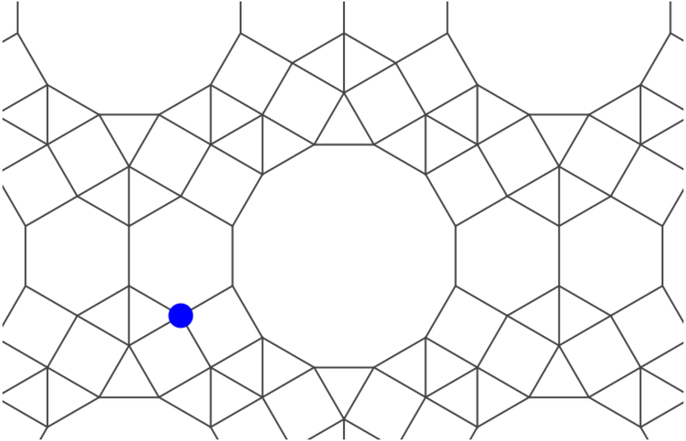
roots of unity

$$\omega^{12} = 1, \quad \omega = e^{\frac{2\pi i}{12}}$$

$$\omega^{n} = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$



Vertices as integer linear combinations of basic directions



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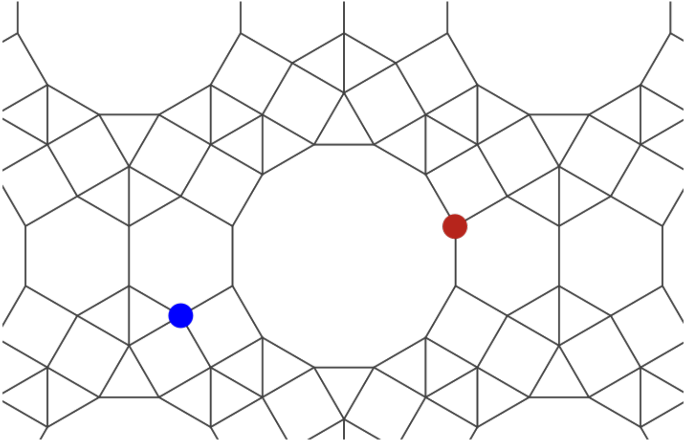

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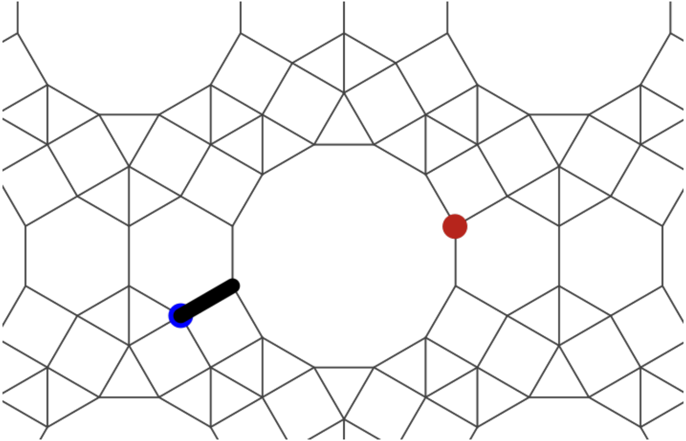
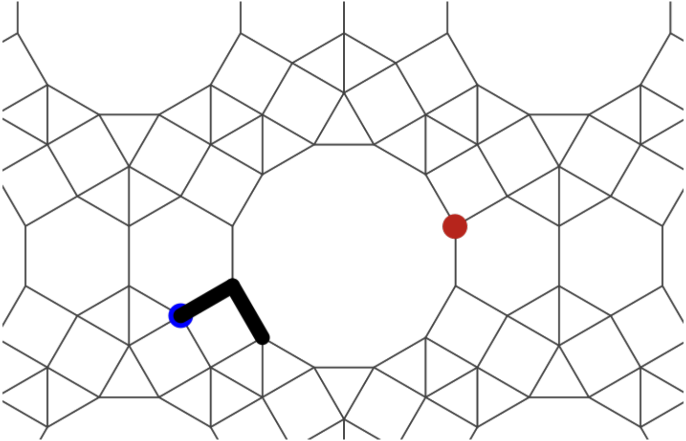


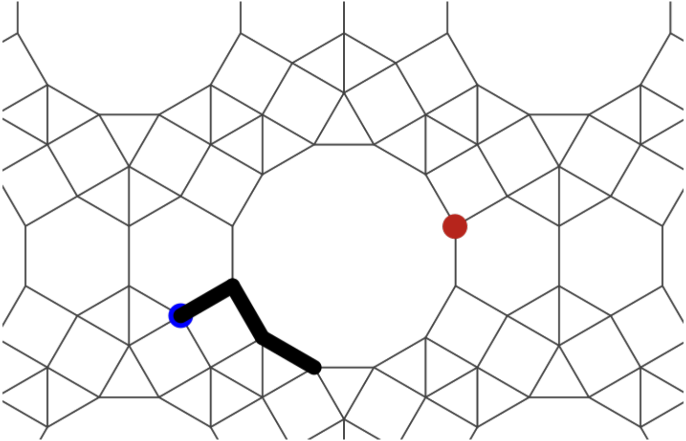
ω

Vertices as integer linear combinations of basic directions



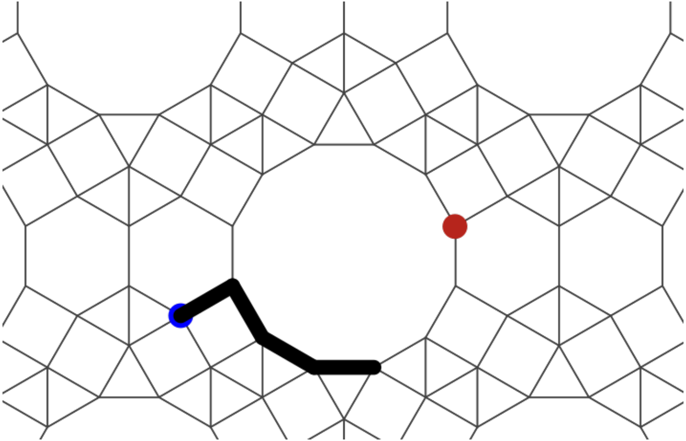
$\omega + \omega^{10}$

Vertices as integer linear combinations of basic directions



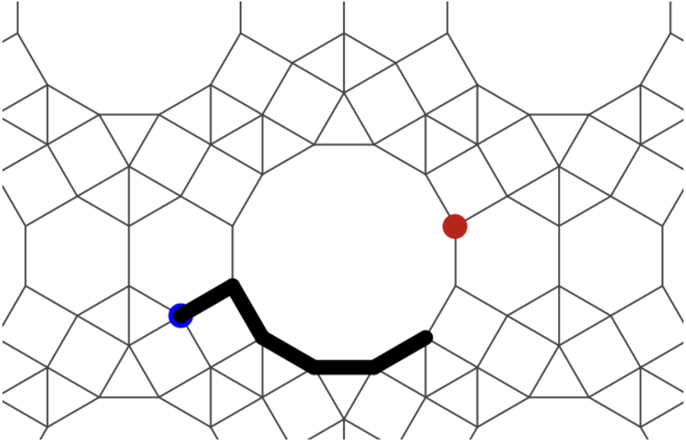
$\omega + \omega^{10} + \omega^{11}$

Vertices as integer linear combinations of basic directions



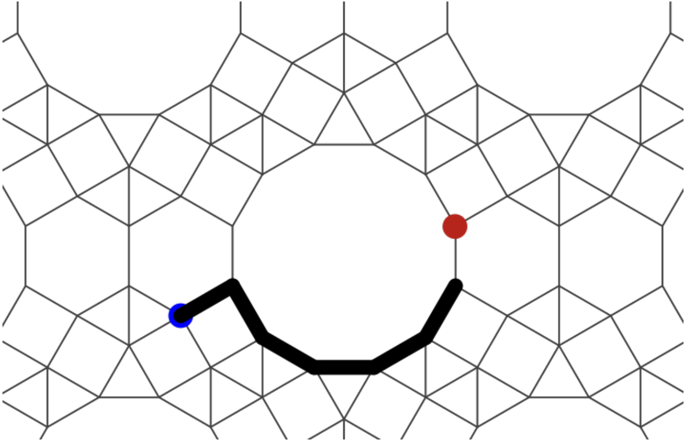
$\omega + \omega^{10} + \omega^{11} + \omega^0$

Vertices as integer linear combinations of basic directions



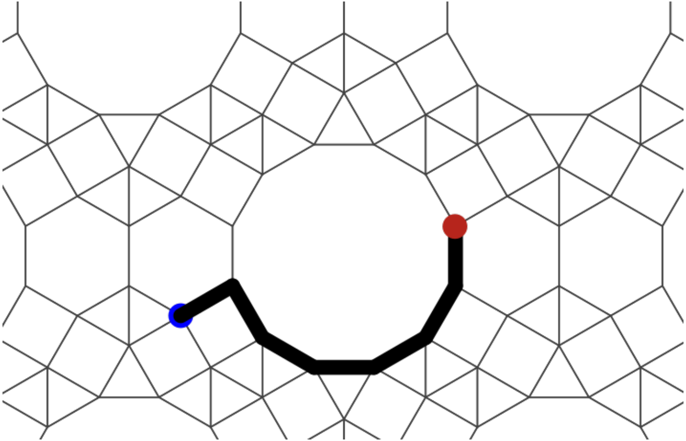
$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega$

Vertices as integer linear combinations of basic directions



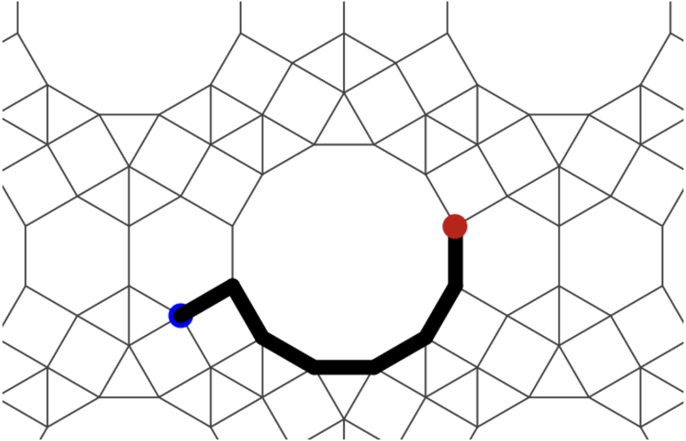
$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2}$

Vertices as integer linear combinations of basic directions



$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3}$$

Vertices as integer linear combinations of basic directions



$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3} = \omega^{11} + \omega^{10} + \omega^{3} + \omega^{2} + 2\omega + 1$$

Tiling symbols

Vertices and translation vectors are expressed in $\mathbb{Z}[\omega]$ = polynomials in ω. Not unique

Polynomials in ω can be reduced mod $\omega^4 - \omega^2 + 1$, minimal polynomial of ω

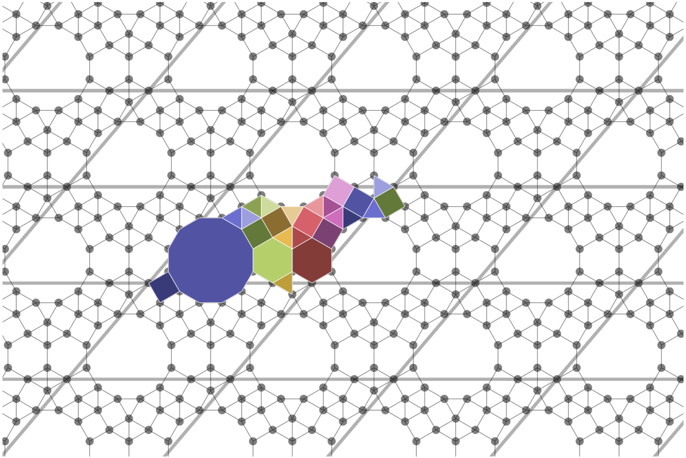
Thus

$$\mathbb{Z}[\omega] = \mathbb{Z}1 + \mathbb{Z}\omega + \mathbb{Z}\omega^2 + \mathbb{Z}\omega^3$$

gives a unique representation!

$$\begin{aligned}
\omega^4 &= -1 + \omega^2 &&= [-1, 0, 1, 0] \\
\omega^5 &= -\omega + \omega^3 &&= [0, -1, 0, 1] \\
\omega^6 &= -1 &&= [-1, 0, 0, 0] \\
\omega^7 &= -\omega &&= [0, -1, 0, 0] \\
\omega^8 &= -\omega^2 &&= [0, 0, -1, 0] \\
\omega^9 &= -\omega^3 &&= [0, 0, 0, -1] \\
\omega^{10} &= 1 - \omega^2 &&= [1, 0, -1, 0] \\
\omega^{11} &= \omega - \omega^3 &&= [0, 1, 0, -1]
\end{aligned}$$

Tiling symbols



0	3	2	1
2	3	−2	−4
0	0	0	0
0	1	0	−1
1	1	−1	−1
0	2	0	−1
0	2	0	0
1	2	−1	−2
1	2	−1	−1
0	2	1	0
2	2	−2	−2
1	3	−1	−2
0	3	1	0
2	3	−2	−3
2	3	−2	−2
0	3	2	0
2	3	−1	−2
1	3	1	0
2	4	−2	−3
2	4	−1	−3
2	4	−1	−2
1	4	1	−1
2	4	0	−2
2	4	0	−1
2	5	−1	−3
2	5	0	−3

Representation of periodic tilings by regular polygons

Soto Sánchez (2020)

Geometry

- lattice coordinates

$$v = [a_0, a_1, a_2, a_3]$$

$$v = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3$$

- vertices stored in hash table

$$V[a_0, a_1, a_2, a_3]$$

Topology

- implicit reconstructed from nearest neighbors found in constant time

Our vertex-centric representation for adaptive diamond-kite meshes

Geometry

- 3-adic lattice coordinates

$$v = [a, b, m]$$

base triangular mesh

$$v = \frac{1}{3^m}(a + b\omega^2)$$

barycenters

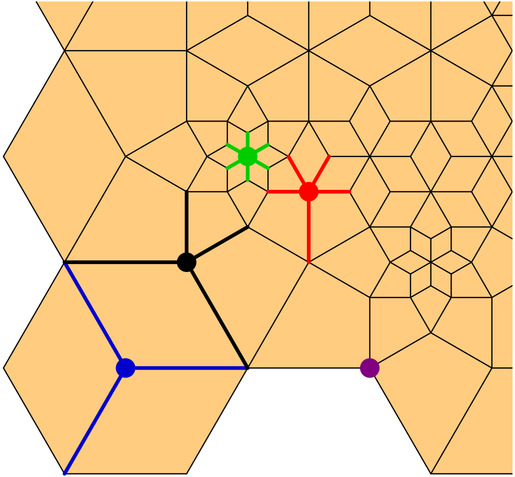
- vertices stored in hash table

$$V[a, b, m]$$

Topology

- implicit reconstructed from type, orientation, scale of vertex stars in constant time

Our vertex-centric representation for adaptive diamond-kite meshes



v	a	b	m	d	k	n
blue	0	0	0	3	0	0
black	0	1	0	4	6	0
red	2	5	1	5	9	1
green	0	2	0	6	1	3
purple	2	0	0	0	0	0

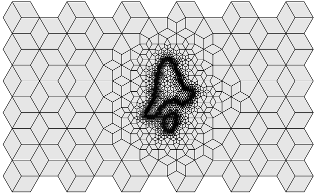
geometry (a, b, m) — topology (d, k, n)

Our vertex-centric representation – CSV

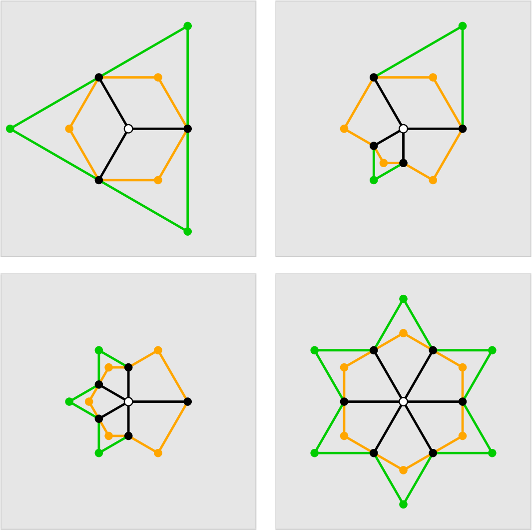
```
a,b,m,d,k,n
0,0,0,3,0,0
1,0,0,0,0,0
-1,1,0,0,0,0
0,-1,0,0,0,0
0,1,0,3,2,0
-1,0,0,0,0,0
1,-1,0,0,0,0
1,1,0,3,0,0
3,0,0,3,0,0
...
136,188,3,3,4,6
136,189,3,3,6,6
135,190,3,3,8,6
134,190,3,3,10,6
139,179,3,3,4,6
139,180,3,3,6,6
138,181,3,3,8,6
137,181,3,3,10,6
137,180,3,3,0,6
138,179,3,3,2,6
```

```
OBJ
3559 3525 0
v 0.0 0.0 0
v 1.0 0.0 0
v -0.5 0.866025403784 0
v -0.5 -0.866025403784 0
v 0.5 0.866025403784 0
v -1.0 0.0 0
v 0.5 -0.866025403784 0
v 1.5 0.866025403784 0
v 3.0 0.0 0
...
f 281 1127 1086 1128
f 278 936 921 937
f 1235 1330 1339 1248
f 1242 1245 1340 1244
f 573 2484 2486 2485
f 98 1452 317 1453
f 2995 3079 3073 3074
f 177 178 180 179
...
```

```
OFF
3559 3525 0
0.0 0.0 0
1.0 0.0 0
-0.5 0.866025403784 0
-0.5 -0.866025403784 0
0.5 0.866025403784 0
-1.0 0.0 0
0.5 -0.866025403784 0
1.5 0.866025403784 0
3.0 0.0 0
...
4 280 1126 1085 1127
4 277 935 920 936
4 1234 1329 1338 1247
4 1241 1244 1339 1243
4 572 2483 2485 2484
4 97 1451 316 1452
4 2994 3078 3072 3073
4 176 177 179 178
...
```

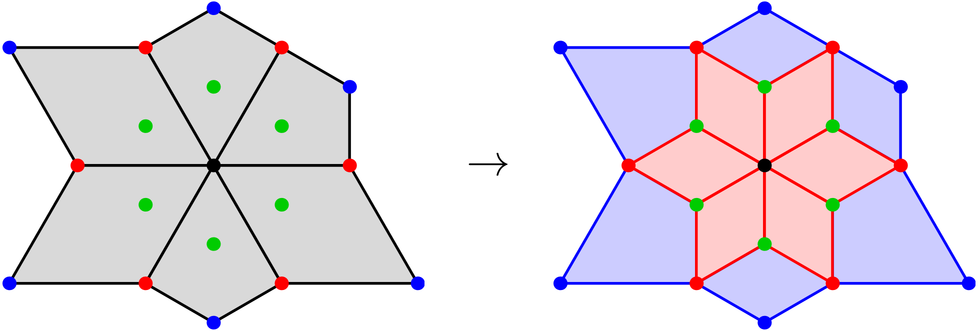


Our vertex-centric representation – standard stars



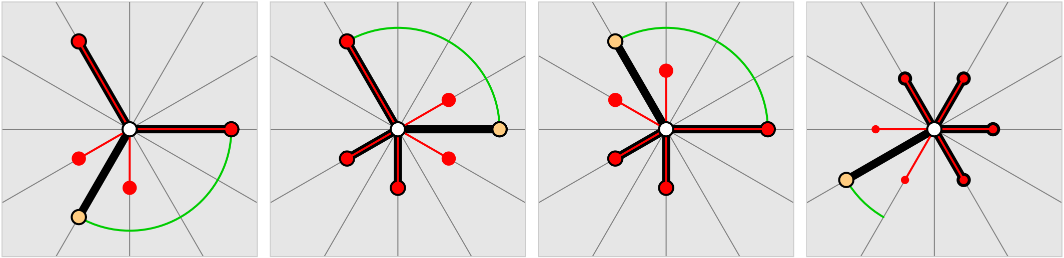
degree	adjacent	opposite1	opposite2
3	[1, 0, 0]	[0, 1, 0]	[0, 2, 0]
	[−1, 1, 0]	[−1, 0, 0]	[−2, 0, 0]
	[0, −1, 0]	[1, −1, 0]	[2, −2, 0]
4	[1, 0, 0]	[0, 1, 0]	[0, 2, 0]
	[−1, 1, 0]	[−1, 0, 0]	
	[−1, −1, 1]	[0, −2, 1]	[0, −1, 0]
	[1, −2, 1]	[1, −1, 0]	
5	[1, 0, 0]	[0, 1, 0]	
	[−1, 2, 1]	[−2, 2, 1]	[−1, 1, 0]
	[−2, 1, 1]	[−2, 0, 1]	[−1, 0, 0]
	[−1, −1, 1]	[0, −2, 1]	[0, −1, 0]
	[1, −2, 1]	[1, −1, 0]	
6	[1, 0, 0]	[2, 2, 1]	[1, 1, 0]
	[0, 1, 0]	[−2, 4, 1]	[−1, 2, 0]
	[−1, 1, 0]	[−4, 2, 1]	[−2, 1, 0]
	[−1, 0, 0]	[−2, −2, 1]	[−1, −1, 0]
	[0, −1, 0]	[2, −4, 1]	[1, −2, 0]
	[1, −1, 0]	[4, −2, 1]	[2, −1, 0]

Our vertex-centric representation – refinement



update orientations and scales

Our vertex-centric representation – update

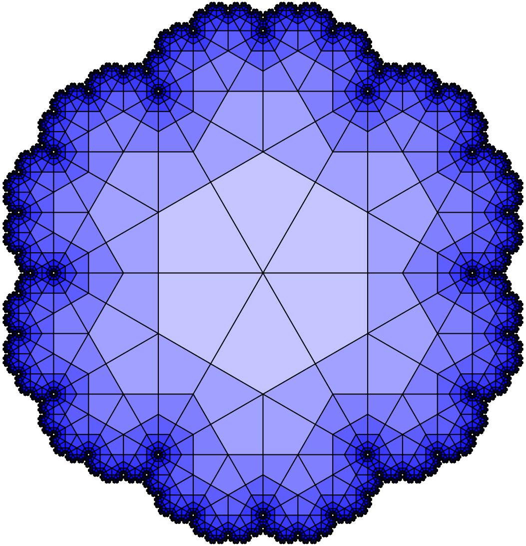


Our vertex-centric representation – features

- **exact** — uses integers for both geometry and topology
- **concise** — much smaller than standard topological data structures
- **highly compressible** — compressed CSV = 20% of compressed OBJ and OFF
- **geometrically meaningful** — vertex stars replace explicit adjacency relations
- **expressive** — topological elements and relations reconstructed in constant time
- **performant** — relies on a good hash table
- **general** — framework for representing general diamond-kite meshes

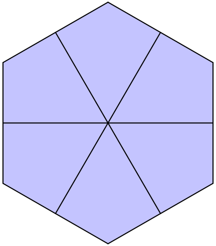
Kite fractals

Fathauer (2001)



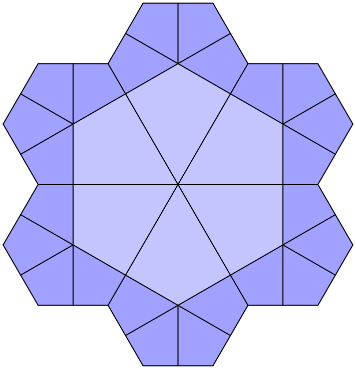
Kite fractals

Fathauer (2001)



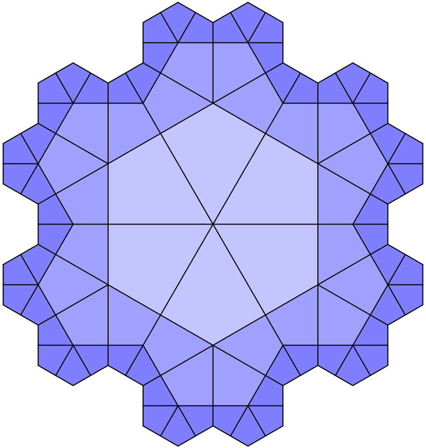
Kite fractals

Fathauer (2001)



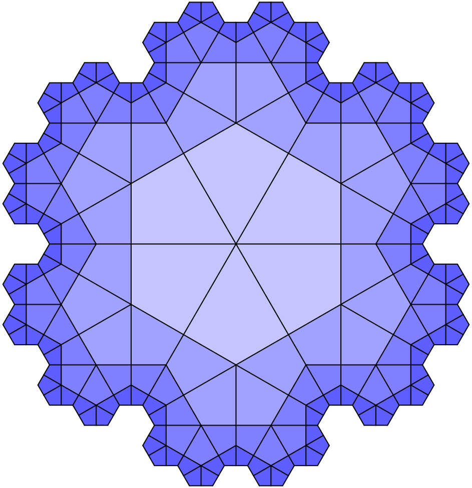
Kite fractals

Fathauer (2001)



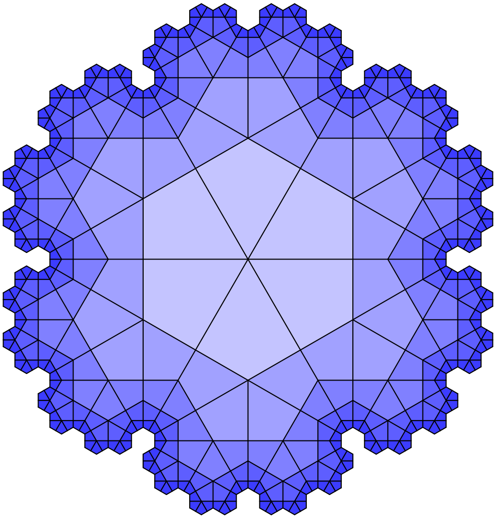
Kite fractals

Fathauer (2001)



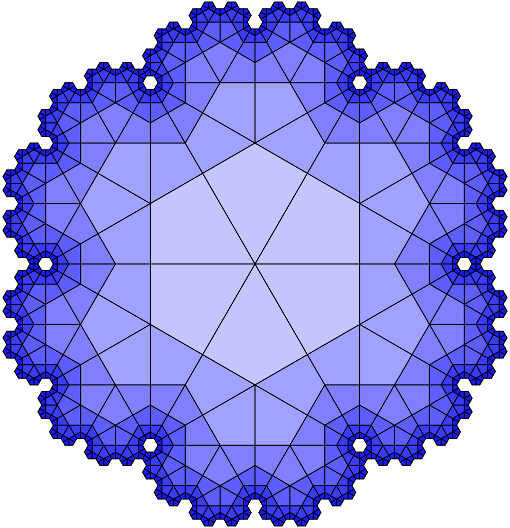
Kite fractals

Fathauer (2001)



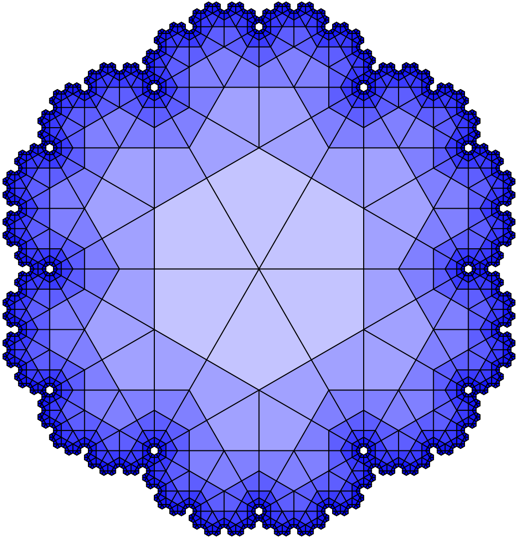
Kite fractals

Fathauer (2001)



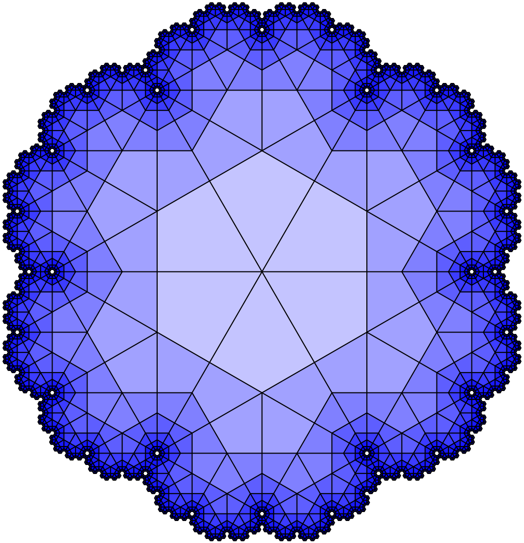
Kite fractals

Fathauer (2001)



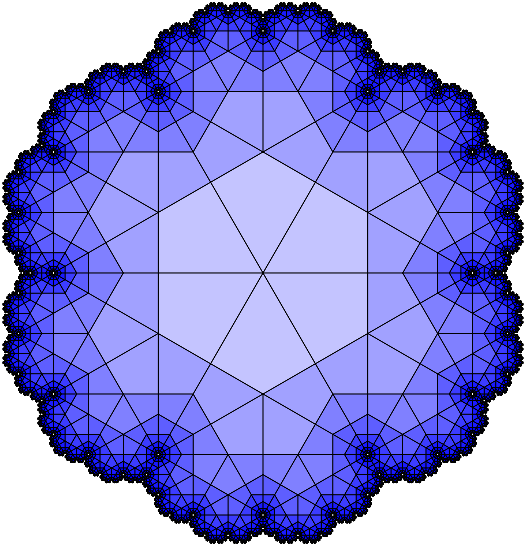
Kite fractals

Fathauer (2001)



Kite fractals

Fathauer (2001)



Kite fractals – vertex-centric representation

Same core

- 3-adic lattice coordinates
- topology represented by type, orientation, and scale of vertex stars
- cloud for storing vertices
- standard stars as templates

Kite fractals – vertex-centric representation

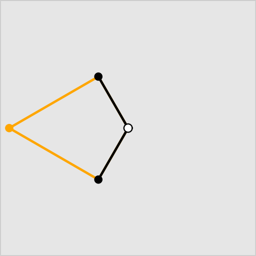
Same core

- 3-adic lattice coordinates
- topology represented by type, orientation, and scale of vertex stars
- cloud for storing vertices
- standard stars as templates

Different details

- refinement happens only at the boundary
- holes appear during refinement
- vertex stars are completely different

Kite fractals – stars



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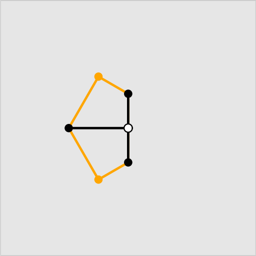

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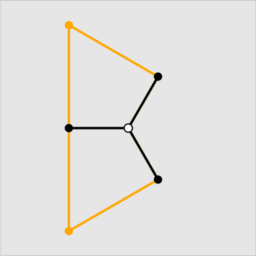

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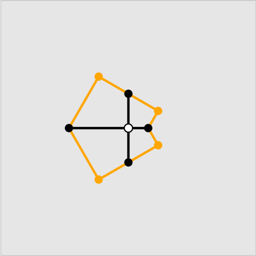

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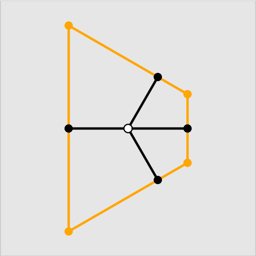

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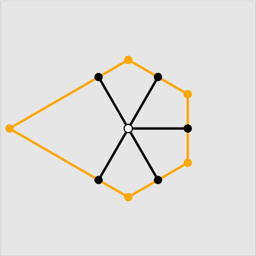
43



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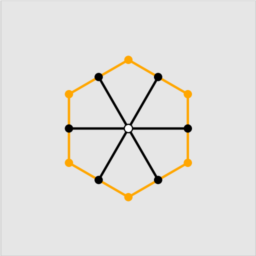

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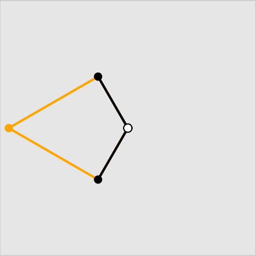
Kite fractals – refinement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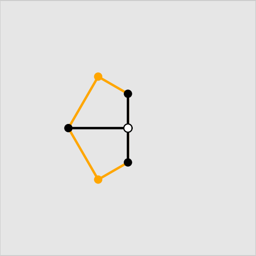
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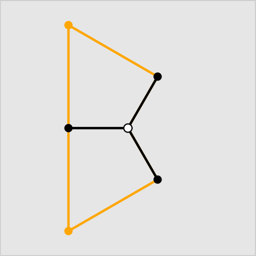
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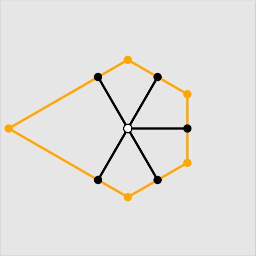
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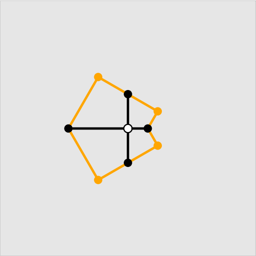
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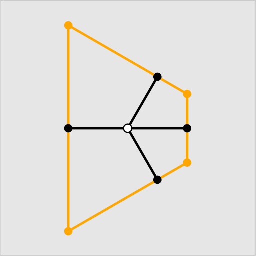
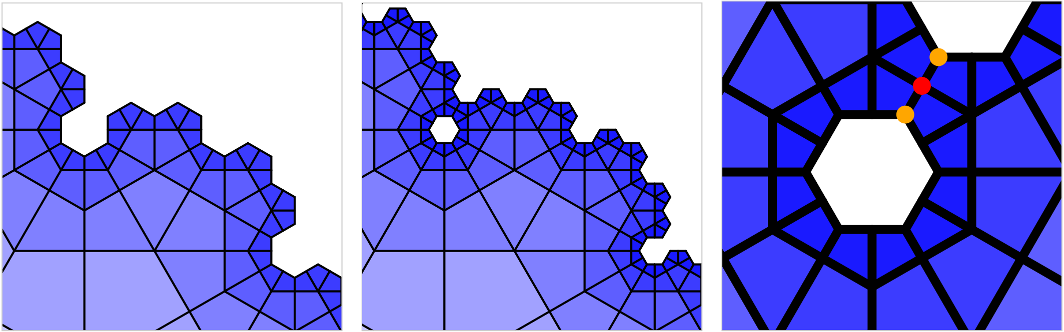
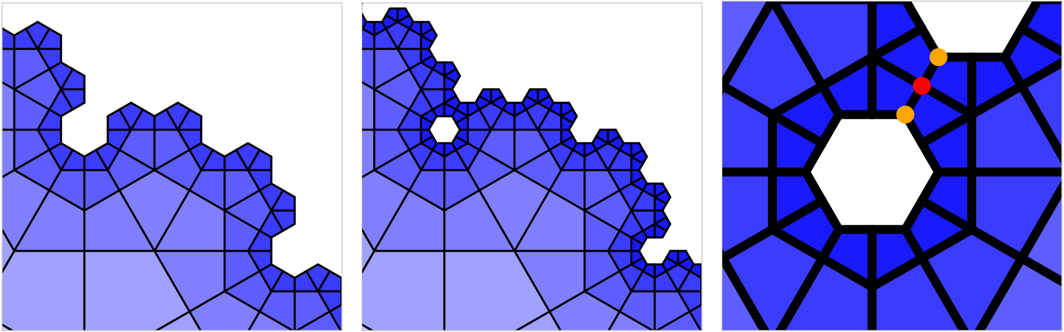

42

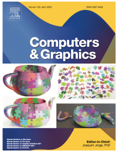
Kite fractals – collisions and holes



Kite fractals – collisions and holes



See also



paper	lhf.impa.br/publications.html
code	github.com/lhf/dk
GRSI	replicabilitystamp.org

A vertex-centric representation for adaptive diamond-kite meshes

Luiz Henrique de Figueiredo

impa

Visgraf Vision and Graphics Laboratory